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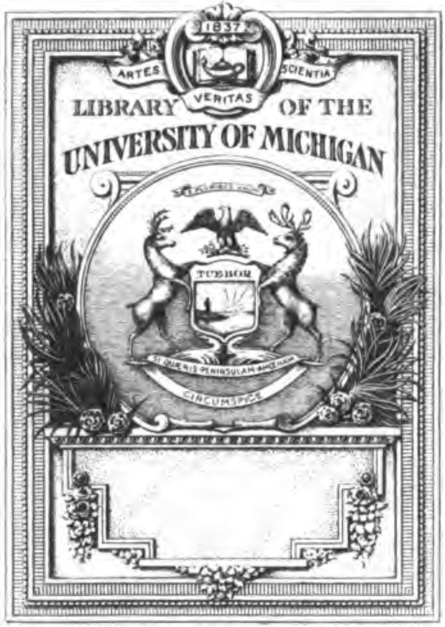

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ELEMENTARY PRACTICAL MATHEMATICS

ELEMENTARY
PRACTICAL MATHEMATICS

BY

FRANK CASTLE, M.I.M.E.

MECHANICAL LABORATORY, ROYAL COLLEGE OF SCIENCE, SOUTH KENSINGTON.
LECTURER IN MATHEMATICS, PRACTICAL GEOMETRY, MECHANICS, ETC.,
AT THE MORLEY COLLEGE, LONDON

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PREFACE.

THE inability of artizan students to profit by much of the technical instruction given in polytechnics and similar institutes is largely due to insufficient preliminary training in Mathematics, for without a fair knowledge of this subject little or no real progress can be made in any branch of Physical Science. Notwithstanding this, classes in pure Mathematics are neither popular nor well attended by students of the industrial class. Even the enthusiastic apprentice seems to find it difficult to connect the instruction usually given in arithmetic, algebra, and geometry with the work of the shop or factory. A number of attempts have been made to remedy this state of affairs. Classes in Technical and Workshop Arithmetic, Machine Calculations, Mensuration, and similar subjects have been started in many centres. The latest attempt to interest and benefit the "practical man" is the syllabus of Practical Mathematics issued by the Department of Science and Art. Judging from the Examination paper of May last, and the lectures delivered by Prof. Perry, F.R.S., to working men and to science teachers, the object of the course is to show the student how to prove and verify truths by experiment instead of arriving at elementary results through the ordinary processes of Euclid and other academic authorities. Abstract reasoning is to be relegated to the background, and concrete facts are to form the basis of the student's work.

The syllabus of Practical Mathematics, to which reference has been made, is so extensive in its scope that it is difficult to deal satisfactorily with all the points in it in a text-book such as the present. The course of study outlined in the following pages will consequently be found more than can be undertaken by backward students in one session. In such cases the plan of completing the course in two years may reasonably be adopted.

Readers familiar with the published works of Prof. Perry, and those who have attended his lectures, will at once perceive how much of the plan of the book is due to his inspiration. But while claiming little originality, the writer has certainly endeavoured to give teachers of the subject the results of a long experience in instructing practical men how to apply the methods of the Mathematician to their everyday work. The exercises, which number close upon 1400, are either original or have been selected after careful consideration from public examination papers.

The writer is glad to take this opportunity of thanking Mr. F. W. Arnold, A.R.C.S., A.M.I.C.E., for reading through the proof sheets, and to express his indebtedness to Prof. R. A. Gregory and Mr. A. T. Simmons, A.R.C.S., B.Sc., for very valuable criticisms and help.

F. CASTLE.

LONDON, *August*, 1899.

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PRACTICAL MATHEMATICS.

CHAPTER I.

MEASUREMENT. UNITS OF LENGTH.

Measurement.—The measurement of a quantity is known when we have obtained a number which indicates its magnitude.

It is necessary, therefore, to select some definite quantity of the same kind, as a unit, and then proceed to find how many times the unit is contained in the quantity to be measured. The number of times that the unit is contained in the given quantity is the numerical value of the quantity.

Nearly all the quantities with which the man of science and the practical man are called upon to deal are concerned either directly or indirectly with **length, mass,** or **time.**

Fundamental and Derived Units.—The primary or fundamental units are three in number, namely :

Unit of length.
„ *mass.*
„ *time.*

All derived units are either multiples or sub-multiples of these primary units.

Units of Length.—In order that length may be measured there must be a *unit* and a *standard.* The unit is a certain definite distance with which all other distances can be compared; and a standard is a bar on which the unit is clearly, accurately, and permanently marked. The two units most generally adopted are the **yard** and the **metre.**

The British System.—In this system the *unit of length* is the *yard.* It may be defined as the distance between two lines on a particular bronze bar when the bar is at a certain temperature (62° F.). The bar is deposited at the Standards Office of the Board of Trade.

The importance of specifying the temperature at which the distance between the two lines on the bar is the exact length, will be evident when the alteration in the size of bodies due to a change of temperature is considered.

Sub-Multiples and Multiples of Yard.—The yard is for many purposes inconvenient, and smaller units, or as they are called *sub-multiples* of the unit of length, are often used. To obtain these smaller units, the yard is divided, first into three equal parts each called a *foot.* The foot is again divided into twelve equal parts called inches; an inch is further subdivided into eight, ten, sixteen (or more) equal parts.

Where fractional parts of a foot are required it is often convenient to express feet and the parts of a foot in a better form than as feet and inches. This is effected by dividing the foot into ten equal parts, the fractional parts are then denoted by $\frac{1}{10}$, $\frac{2}{10}$, etc., or ·1, ·2, etc.

Multiples of the yard are used when comparatively long distances have to be expressed. Thus, 1 mile = 1760 yards = 5280 feet. The multiples and sub-multiples of the unit are given in the following Table.

It must be at once remarked that in the British System there is no simple relation connecting the unit of length with the units of area, volume, and mass. It is only by a slow and troublesome process of multiplication and division such as reducing feet or miles to inches, ounces to pounds, etc., that we can proceed to find areas, volumes, etc.

British Measures of Length.

[The unit is divided by 3 and 36, etc.; also multiplied by 2, $5\frac{1}{2}$, 220, and 1760.]

12 inches = 1 foot.
3 feet = 1 **yard** (unit).
2 yards = 1 fathom.
5½ yards = 1 rod, or pole.

40 poles = 1 furlong.
8 furlongs
1760 yards = 1 mile.
or 5280 feet

The above table will serve to show how very inconvenient the British Unit and its sub-multiples and multiples are for most purposes, since so cumbersome a calculation is required to convert the one into the other.

Fig. 1 shows one end of a 12 inch *steel scale* or straight edge. An inch is represented divided into 16 equal parts, and for the convenience of the person using the scale some of the lines are made longer than others. This enables a dimension to be read off much more quickly and accurately than would otherwise be the case. Thus the cross-line at *c* dividing an inch into two equal parts is seen to be longer than any cross line between *a* and *c*, or between *c* and *b*. In a similar manner the cross lines indicating the quarter inches are longer than those indicating the eighths, these in turn being longer than the sixteenths. As the number of divisions increases the lines naturally become more crowded together, until, as shown, the distances between the divisions when an inch is divided into 32 equal parts, or those indicating the sixty-fourths become very minute.

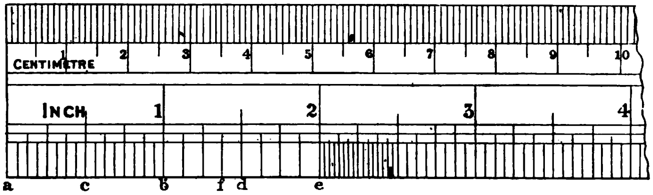

FIG. 1.—Inches divided into 8 and 16 equal parts. A decimetre divided into 10 centimetres and 100 millimetres. (The inches and centimetres are not drawn to true scale, but their comparative dimensions may be seen.)

The French or Metric System.—The Metric System is extensively used for all scientific, and in many cases for commercial purposes, and is in every way better and simpler than the British method.

In the metric system the standard of length is the **metre**, defined originally to be the 10,000,000th part of the length of the earth's meridian from the pole to the equator. The determination of this length was found to be slightly incorrect, and the metre, as in the case of the British standard yard, is best defined as the distance at 0° C. between two marks on a platinum bar preserved at Paris, and known as the Mètre des Archives. Copies of the accurate standard metre are to be found in several national repositories in Europe.

Multiples and Sub-Multiples of Metre.—The metre is divided into 10 equal parts called *decimetres*; the decimetre is divided into 10 equal parts each a *centimetre*: hence a centimetre is one hundredth of a metre, and this sub-multiple of the unit is the most commonly used of the metric measures of length.

The centimetre is divided into 10 equal parts each known as a *millimetre.*

The metre is larger than the yard since it is equal to 39·37 inches, or $3\frac{2}{7}$ feet.

The foot is equal in length to 30·48 centimetres.

Referring to the upper part of the scale in Fig. 1, the division of a decimetre into 10 centimetres (cm.) is represented, but not to true scale. Each centimetre is further subdivided into 10 equal parts, each of which is a millimetre.

It will be seen that a length of 10 cm. is approximately equal to 4 inches. A more accurate relation to remember is that a length of 25·4 centimetres or 254 millimetres is equal to the length of 10 inches. The sub-multiples and multiples of the metre are given in the following table.

Metric Measures of Length.

10 **milli**metres = 1 **centi**met
10 **centi**metres = 1 **deci**metre.
10 **deci**metres = 1 metre.
10 metres = 1 **Deka**metre.
10 **Deka**metres or 100 metres = 1 **Hekto**metre.
10 **Hekto**metres or 1000 metres = 1 **Kilo**metre.

It is convenient to remember that 2·5 cm. = 1 inch (approx.), for this enables a dimension to be readily converted from one system to the other.

It should also be borne in mind that the Latin prefixes *deci-*, *centi-*, *milli-*, are always used for the tenth, hundredth, and thousandth parts of the unit; in a similar manner the Greek prefixes *Deka-*, *Hekto-*, and *Kilo-*, are used for the multiples of the unit.

Conversion of British to Metric Measures of Length.

1 inch = 2·54 centimetres.
1 foot = 30·48 centimetres.
1 yard = 0·914 metre.
1 mile = 1609·3 metres.

Metric to British.

1 millimetre = ·039 inch.
1 centimetre = ·394 inch.

$$1 \text{ metre} = \begin{cases} 39{\cdot}371 \text{ inches.} \\ 3{\cdot}28 \text{ feet.} \\ 1{\cdot}094 \text{ yards.} \end{cases}$$

1 Kilometre = 0·621 mile.

Abbreviations.—The following abbreviations are usually used, and should be carefully remembered; this may be easily effected by taking the precaution to use the abbreviations on all possible occasions.

Length.

in.	is used	to denote	inch or inches.
ft.	„	„	feet.
dcm.	„	„	decimetre or decimetres.
cm.	„	„	centimetre or centimetres.
mm.	„	„	millimetre or millimetres.
gm.	„	„	gram or grams.

Feet and inches are also indicated by the uses of dashes, ′ and ″, at the top and right-hand side of a figure.

Thus, a dimension of 2 feet and 4 inches may be written as 2 ft. 4 in., or 2′ 4″. This method, though widely adopted in practical work, may (unless care is exercised) sometimes be confusing, for the signs ′ and ″ are also used to designate certain angular measurements.

Summary.

Measurement.—A convenient unit is selected, and the number of times the unit is contained in the quantity to be measured is the numerical value of the quantity.

The three quantities to be measured are **Length**, **Mass**, and **Time**.

Primary or fundamental units are comprised in **unit of length**; **unit of mass**; and **unit of time**.

The **derived units** are *multiples, or sub-multiples* of the fundamental units.

The **yard** and the **metre** are the two *fundamental units of length.*

In the *British System*, the *unit of length*, the **yard**, is *the distance between two marks on a certain bronze bar when the bar is at a temperature of* 62° F. The bar is deposited in the Office of the Standards.

The **derived units** are obtained by dividing the yard into three equal parts, each **1 foot**, which is again subdivided into 12 equal parts, called inches; these are again divided into 8, 10, 12, 16, or more equal parts.

Multiples of the unit are 1 mile = 1760 yards, or 5280 feet.

In the French or **Metric System** the *unit of length* is the **Metre**, and is *the distance between two marks on a platinum bar when the temperature of the bar is* 0° C.

The *derived units* are obtained by dividing the metre into 10 equal parts each called a **decimetre**; this is further subdivided into 10 equal parts or **centimetres**, and again into 10, giving **millimetres.**

CHAPTER II.

UNITS OF AREA AND VOLUME. DENSITY. RELATIVE DENSITY. PRINCIPLE OF ARCHIMEDES.

Unit of Area.—Measurement of area, or square measure, is derived from and calculated by means of Measures of Length. Thus the *unit of area is the area of a square*, the side or edge of which is the unit length.

Area of a Square Yard, or Unit Area.—If the unit length be a yard proceed as follows: Make AB equal to 3 feet as in Fig. 2, and upon AB construct a square. Divide AB and BC ineah cto three equal parts, and draw lines parallel to AB and

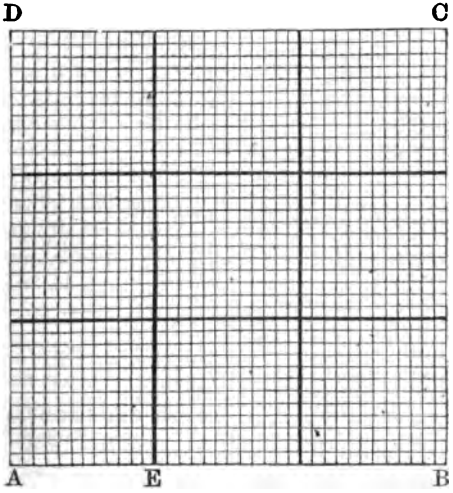

Fig. 2.—1 square yard equals 9 square feet, or 9×144 square inches.

BC as in the figure. The unit area is thus seen to consist of nine smaller squares, every side of which represents a foot; thus the unit area, the square yard, contains 9 square feet. The smaller measures of length, the foot and the inch, are much more generally used than the yard. If the unit of length *AE* (Fig. 2) be one *foot*, the unit of area *AEF* is one square foot. In a similar manner, when the unit of length is one inch, the unit of area is one square inch. If the unit of length be 1 centimetre, the unit of area is one square centimetre. (Fig. 3.)

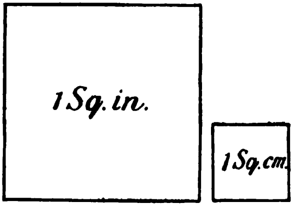

FIG. 3.—Square inch and square centimetre.

If the edge of the square on *AE* (Fig. 2) represent, on some convenient scale, 1 foot, then by dividing *AE* and *AF* each into twelve equal parts, the distance between consecutive divisions would denote an inch. If through these points lines be drawn parallel to *AE* and *AF* respectively, it will be found that there are 12 rows of squares parallel to *AE* and 12 squares in each of these 12 rows. Hence, the area of a square foot represents 144 square inches. When comparatively large areas, such as the area of a field, have to be estimated, the measurements of length, or linear measurements, are made by using a chain 22 yards long. Such a chain is subdivided into 100 links. The square measurements, or areas, are estimated by the *square chain*, or 484, (22 × 22) square yards in area. Or, the area of a square, the edge of which is 22 yards, will be (since each chain consists of 100 links) 100 × 100 = 10,000 sq. links. Hence we have the relation :

1 chain = 22 yards = 100 links.
1 square chain = 484 square yards = 10,000 sq. links
10 square chains = 4840 square yards = 1 acre.

Metric Measures of Area.—As the metric unit of length is the metre, the unit of area (Fig. 4) is a square *ABDE*, having the length of its edge equal to 1 metre, and its area consequently equal to 1 square metre.

If *AB* and *BD* are each divided into ten equal parts and lines drawn parallel to *AB* and *BD*, as shown, the unit area is divided into 100 equal squares, each of which is a square decimetre.

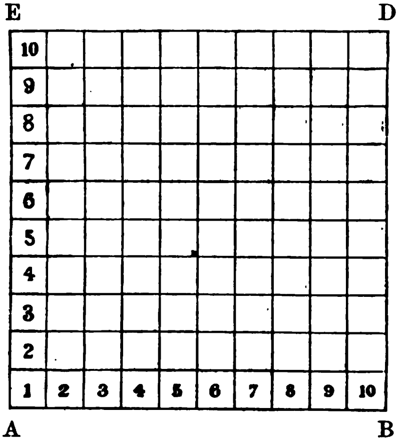

Fig. 4.—Representing a square metre divided into 10 decimetres. Scale $\frac{1}{20}$.

In scientific work the **centimetre** is the unit of length usually selected, and the unit of area is then one **square centimetre.**

British Measures of Area or Surface.

[*Unit area* = 1 *square yard.* Larger and smaller units obtained by multiplying by 4840 and dividing by 27 and 3888.]

144 square inches = 1 square foot.
1296 square inches or 9 sq. ft. = 1 square yard.
4840 square yards = 1 acre.
640 acres = 1 square mile.

Metric Measures of Area.

100 square millimetres = 1 square centimetre.
10000 „ „ = 100 sq. cm. = 1 sq. decim.
100 „ decimeters = 1 square metre.

Conversion Table.

British to Metric.

1 sq. in. $=6·451$ sq. cm.
1 sq. ft. $=929$ sq. cm.
1 sq. yard $=8361·13$ sq. cm.
1 acre $=4046·7$ sq. metres.
1 sq. mile $=2·59$ sq. Km.
$=2·59 \times 10^{10}$ sq. cm.

Metric to British.

1 sq. cm. $=$ 0·155 sq. in.
1 sq. m. $=10·764$ sq. ft.
1 sq. m. $=$ 1·196 sq. yard.
1 sq. Km. $\doteq 0·3861$ sq. mile.

Units of Volume.—In the British system an arbitrary unit, the **gallon,** is the *standard unit of volume*, and is defined as the volume occupied by 10 lbs. of pure water at a temperature of 62° F.

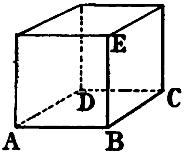

FIG. 5.—Cubic foot. Scale $\frac{1}{24}$.

A larger unit is the capacity of a cube the base of which, *ABCD* (Fig. 5), is a square the side of which is 1 foot and the height *BE* also 1 foot. The volume of such a cube is 1 cubic foot. As the base contains 144 square inches, the volume is 1728 cubic inches.

As already mentioned in the case of the unit of length, with one or two exceptions, bodies expand by heat and contract when cooled; hence the temperature of the water must be given. The weight of a cubic foot of water depends upon its temperature.

Thus at 32° F. the weight is 62·418 lbs., at 62° F. the weight is 62·355 lbs., and at 212° is 59·64 lbs. A good average value is 62·3 lbs. For convenience in calculations a cubic foot is sometimes taken to be $6\frac{1}{4}$ gallons, its weight 1000 oz., or 62·5 lbs. Hence the weight of a pint is about $1\frac{1}{4}$ lbs.

When the unit of length is *one inch*, the unit of volume is one *cubic inch*.

In Fig. 6 a cubic yard is shown. If *ABCD* represent a square having its edge 1 yard, the area of the square is 9 square feet. If the vertical sides, one of which is shown at *DE*, be divided into three equal parts, and the remaining lines be drawn parallel to *DE* and the base respectively. Then, as will be seen from the figure, there are nine perpendicular rows of small cubes, the sides being 1 foot in length, area of base 1 square foot, and volume 1 cubic foot. Also there are three of these cubes in each row, making in all $3 \times 9 = 27$. Thus 1 cubic yard = 27 cubic feet, *i.e.* $3 \times 3 \times 3 = 27$, and the weight of a cubic yard of pure water would be 27 × 62·3 lbs. = 1682·1 lbs.

In this example and also in considering the weight of a gallon, the student should notice that the specification "pure" water is necessary, for if the water contains matter either in solution or mixed with it, its weight would be altered. Thus, the weight of a cubic foot of salt water is 64 lbs., and the weight of a gallon of muddy water may be 11 or 12 lbs. instead of 10 lbs.

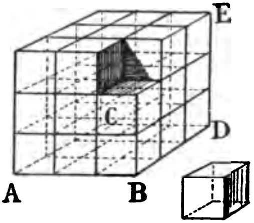

FIG. 6.—Showing a cubic yard and a cubic foot.

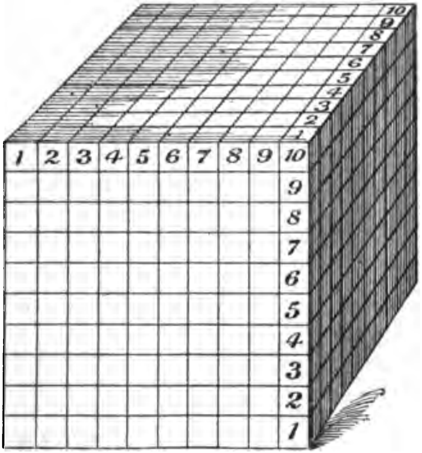

FIG. 7.—Cubic decimetre (1000 cubic centimetres) holds 1 kilogram or 1000 grams (1 litre) of water at 4° C.

Metric System.—In the Metric System the unit of volume is the volume occupied by a **Kilogram** of pure water at 4° C. and is called a **Litre.** It is very nearly a cubic decimetre or 1000 cubic centimetres (Fig. 7), and is equal to 1·76 English pints.

We have found that the unit of area is, for convenience, taken to be one square centimetre, and the corresponding unit of volume will be the cubic centimetre (c.c.).

MEASURES OF VOLUME OR CAPACITY.

British.

Unit Volume is the volume occupied by **a gallon**, or 10 lbs. of water at a temperature of 62° F.

One fourth part of a gallon is a quart, and an eighth part of a gallon is a pint.

1728 cubic inches = 1 cubic foot. 27 cubic feet = 1 cubic yard.

Metric.

Unit Volume. The **Litre** is the volume occupied by 1 Kilogram of water at 4° C. 1000 cubic centimetres = 1 litre.

The standard temperature 4° C. is not very convenient, and the temperature 62° F., or 16⅔ C., may be used instead; a suitable correction for expansion must be made if necessary.

Volume.

Conversion Table.

1 cub. in.	=16·387	cub. cm.	1 c.cm.	=·061	cub. in.
1 „ ft.	=28316	„	1 litre	=61·027	„
1 „ yard	=764535	„		=1·76	pint, or,
1 pint	=567·63	„		=·22	gallon.
1 gallon	=4541	„			

Units of Mass and Weight.—The mass of a body is the quantity of matter it contains.

The **British Unit of Mass** is the **Pound Avoirdupois.** The mass of a body is ascertained by comparing it with the Standard.

The British Pound is defined as the quantity of matter in a platinum cylinder (Fig. 8) deposited in the offices of the Board of Trade. Accurate copies of the Standard, as well as multiples and sub-multiples of it, are to be found in various parts of the country. The sub-multiples of the unit are obtained by dividing the unit into sixteen equal parts, each called an **ounce**. A smaller measure of weight is obtained by dividing the pound avoirdupois into 7000 equal parts, each of which is called a **grain**. Thus, one-sixteenth part of the unit is an ounce, and one seven-thousandth part a grain.

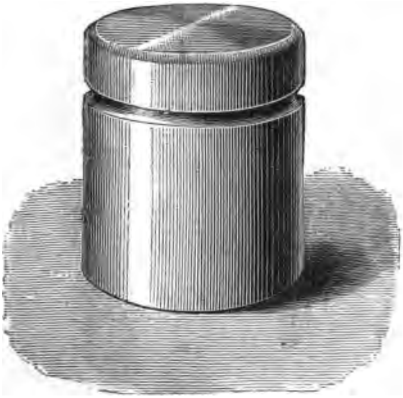

FIG. 8.—Imperial Standard Pound. From Aldous's *Course of Physics* (Macmillan).

By the operation of weighing very accurate copies of the standard can be made. The choice of platinum as a suitable material for the standard is made on account of its being a substance not liable to be affected by atmospheric influences.

The Metric Unit of Mass is the Kilogram.—The metric standard of mass is conveniently obtained from the corresponding unit of volume. The *kilogram*, for such the standard is called, was originally defined to be the quantity of matter in a cubic decimetre, or **litre**, of pure water, at its temperature of maximum density 4° C. It is now defined to be the mass of a platinum cylinder (Fig. 9) deposited in the French Archives. For all practical purposes a *litre* of pure water at 4° C. weighs 1 kilogram or 1000 grams. Hence, since, as has been seen, the litre contains 1000 cubic centimetres (c.c.) a *gram is the weight of a cubic centimetre of water at a temperature of* 4° C. The gram is the usual unit used, and hence when we know the volume in cubic centimetres of any quantity of water at 4° C., we know also its weight in grams.

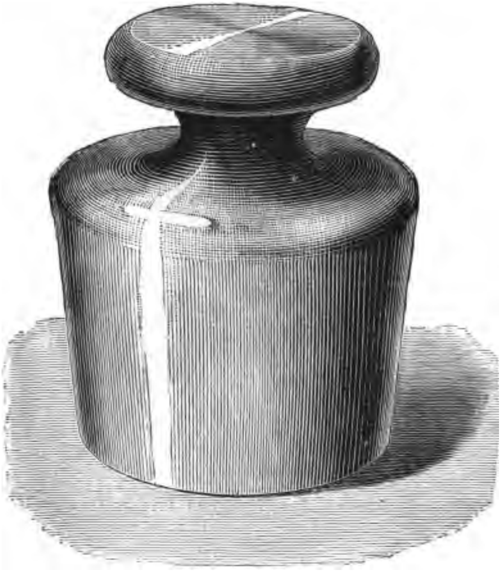

Fig. 9.—The Kilogram. From Aldous's *Course of Physics* (Macmillan).

It is advisable to remember that there are 453·59 grams in a pound; that 1 gram = 15·432 grains; and a kilogram = $2\frac{1}{5}$ lbs.

British and Metric Measures of Mass.

1 grain	= ·0648 gm.	1 gram	= 15·43 grains.
1 ounce avoirdupois	= 28·35 gm.	1 kilo	= 2·2 lbs.
7000 grains 1 pound (lb.)	= 453·59 gm.		
1 ton	= $1·01605 \times 10^6$ gm.		

10 milligrams = 1 centigram.
10 centigrams = 1 decigram.
1000 grams = 1 kilogram.

Weight.—The *weight of a body* is the *attractive* force which the earth exerts on the body at or near its surface. It varies with the position of the body on the earth's surface, being greatest at the poles and least at the equator.

Hence the *weight of a pound* is a quantity of force, viz., the force equal to the tension of a string supporting a mass of 1 lb., or, as it is commonly called, a pound weight. The force of gravity varies with the distance from the earth's centre; thus, at a place where gravity is nil, such as at the centre of the earth, the weight would also be nil.

The weight of the body, equal to the tension of the string supporting it, could be ascertained by attaching the string to a spring balance. If the mass attached to the balance could be carried from the pole to the equator, although the mass remains constant, its weight, as indicated by the balance would vary continuously.

By means of an *ordinary balance* or *pair of scales* the mass of a body in terms of the unit mass, or the weight of a body in terms of the weight of the unit mass, can be obtained at the particular place where the estimation is made.

The operation of finding the weight of a body is called *weighing*. Thus the body whose weight is to be found is placed in one pan of a balance, and *known weights* are placed in the other pan until the two are balanced; the sum of the *weights* used is the *weight* of the body.

Density.—*The density of a substance is* **the mass of the unit volume of it**. Assuming the density to be uniform, the density of a substance, when the unit of mass is one pound and the unit of volume one cubic foot is the number of pounds in a cubic foot of the substance.

In the cases where metric units are adopted, the density is the number of grams in a cubic centimetre of the substance.

Density of Water.—It has been found by means of careful experiments that water does not contract regularly down to the freezing point, but begins to expand before it reaches this temperature. The temperature at which a given mass of water occupies the smallest volume is 39·2° on the Fahrenheit Scale, or 4° on the Centigrade; both below and above. this temperature its volume increases, hence at this temperature water has its *maximum density*, and the weight of a cubic foot is nearly 1000 oz. or 62·5 lbs. The weight of a cubic centimetre of water at its maximum density is 1 gram. It is a matter of common experience that the weights of equal volumes of different substances are not the same. For example, three balls or cubes of the same volume, but made of wood, iron, and lead respectively, would, when placed in a balance, have different weights. The wooden ball is comparatively light in comparison with the heavier iron, and the lead the heaviest of all three.

A vessel containing a cubic foot of water, the mass of which is 62·3 lbs., would, when the water is replaced by the same volume of mercury, contain 13·596 × 62·3 lbs.; the weight of 1 cubic centimetre of water is 1 gram, the weight of the same volume of mercury is 13·596 grams.

Relative Density.—*The relative density of a substance is the ratio of its weight to the weight of an equal volume of a standard substance.** It is obvious that the standard substance should be at any place easily obtainable and in a pure state; pure distilled water fulfils these conditions. The only precaution necessary is with regard to temperature. On account of the expansion and contraction due to heating and cooling, the weight of a cubic unit will depend on its temperature, hence the standard temperature is 4° C.

The *relative density* is usually called the **specific gravity**; the specific gravities of various substances are tabulated in Table III. Thus, if the specific gravity of cast iron is 7·22, then the weight of a cubic foot is 7·22 times the weight of a cubic foot of water $= 7{\cdot}22 \times 62{\cdot}3 = 450$ lbs.

The weight of a cubic centimetre of cast iron will evidently be 7·2 grams.

The weight of V cubic feet of water $= Vw$, where w is the weight of unit volume.

Hence, if V denote the volume of a body in cubic feet, and S its specific gravity, the weight of the body is $VS \times w$ (1). If w be the weight of unit volume, then the weight of the body is $VSw = VS \times 62{\cdot}3$ lbs. [Eq. (1)]. In this manner it is customary to define *specific gravity as the ratio of the weight of a given volume of a substance to the weight of the same volume of water.*

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* If necessary portions of this section, more especially those examples dealing with area and volume, may be postponed until the chapters on area and volume have been read.

If the volume of the body is obtained in cubic inches then in Eq. (1) w will denote the weight of one cubic inch (the weight of one cubic inch of water $= 62{\cdot}3 \div 1728 = {\cdot}036$ lbs.).

The method of obtaining the specific gravity of a solid (not soluble in water) depends on what is known as the "Principle of Archimedes": **When a body is immersed in a liquid it loses weight equal to the weight of the liquid which it displaces**; that is, if the weight of a body is obtained, first in air, and next when immersed in water, the difference in the weights is the weight of an equal volume of water:

$$\therefore \text{ specific gravity of body} = \frac{\text{weight in air}}{\text{weight in air} - \text{weight in water}}.$$

Ex. 1. A piece of ivory weighing 6 lbs. is found to weigh 2·8 lbs. when immersed in water. Find its specific gravity.

$$\text{specific gravity} = \frac{6}{6 - 2{\cdot}8} = 1{\cdot}875.$$

Ex. 2. A piece of metal weighs 62·63 grains in air and 56 grains in water. Find its specific gravity.

$$\text{S.G.} = \frac{62{\cdot}63}{62{\cdot}63 - 56} = 9{\cdot}4.$$

Ex. 3. A piece of metal of specific gravity 9·8 weighs in water 56 grains. What is its true weight?

Let w denote its true weight.

Then
$$9{\cdot}8 = \frac{w}{w - 56};$$

$$\therefore\ 9{\cdot}8w - 9{\cdot}8 \times 56 = w;$$

$$\therefore\ w = \frac{9{\cdot}8 \times 56}{8{\cdot}8} = 62{\cdot}36 \text{ grains}.$$

Ex. 4. If 28 cubic inches of water weigh a pound, what will be the specific gravity of a substance, 20 cubic inches of which weigh 3 lbs.?

As 20 cubic inches weigh 3 lbs., 1 cub. in. $= \frac{3}{20}$;

$$\therefore\ 28 \text{ cub. in.} = 28 \times \tfrac{3}{20} \text{ lbs.} = 4\tfrac{1}{5} \text{ lbs.}$$

But the same volume of water weighs 1 lb.;

$$\therefore\ \text{specific gravity} = \frac{4\frac{1}{5}}{1} = 4\tfrac{1}{5}.$$

Ex. 5. A piece of round steel wire 12 inches long weighs 0·65 lbs., its specific gravity is 7·8, find the area, also the diameter of the wire.

If A denote the area of the wire in square inches,

$$\text{Volume in cubic inches} = 12 \times A.$$

$$\text{Weight} = 12A \times 7{\cdot}8 \times {\cdot}036, \text{ but this is equal to } {\cdot}65 \text{ lbs.};$$

$$\therefore\ 12A \times 7{\cdot}8 \times {\cdot}036 = {\cdot}65;$$

$$\therefore\ A = \frac{{\cdot}65}{12 \times 7{\cdot}8 \times {\cdot}036} = {\cdot}1930 \text{ sq. in.}$$

From this (p. 244) the diameter is found to be $\frac{1}{2}$ inch.

When the specific gravity of a body lighter than water is required, what is called a *sinker* is used, *i.e.* a piece of lead or metal of such a size as to secure complete immersion of the body.

Ex. 6. A piece of wood which weighs 10 lbs. is attached to a piece of metal weighing 20 lbs.; the whole weighs 14 lbs. in water; the metal alone weighs 18 lbs. in water. Find the specific gravity of the metal and the wood.

$$\text{S.G. of metal} = \frac{20}{20-18} = 10; \quad \text{S.G. of wood} = \frac{10}{10+18-14} = {\cdot}714.$$

Ex. 7. Three pints of a liquid of specific gravity ·6 are mixed with four pints of a liquid of specific gravity ·81; if there is no contraction, find the specific gravity of the mixture.

(A pint of water weighs $1\frac{1}{4}$ lbs.)

Weight of 3 pints of liquid of specific gravity ·6

$$= 3 \times 1\tfrac{1}{4} \times {\cdot}6 = 2{\cdot}25 \text{ lbs.}$$

Weight of 4 pints of liquid of specific gravity ·81

$$= 4 \times 1\tfrac{1}{4} \times {\cdot}81 = 4{\cdot}05 \text{ lbs.}$$

$$\text{Hence the weight of the 7 pints} = 2{\cdot}25 + 4{\cdot}05 = 6{\cdot}3 \text{ lbs.}$$

$$\text{But the weight of 7 pints of water} = 7 \times 1\tfrac{1}{4} = 8{\cdot}75 \text{ lbs.}$$

$$\therefore\ \text{specific gravity of mixture} = \frac{6{\cdot}3}{8{\cdot}75} = {\cdot}72.$$

The principle of Archimedes applies to all bodies whether they are heavier or lighter than the liquid in which they are immersed. Thus, *a floating body displaces a volume of water equal in weight to the weight of the body itself.*

Hence $$\text{S.G.} = \frac{\text{volume of body immersed}}{\text{whole volume}}.$$

EXERCISES. I.

1. A piece of lead weighs 3 lbs. 12 oz., and a piece of oak weighs 2 lbs. 8 oz.: what ratio does the mass of the piece of lead bear to the mass of the piece of oak?

2. The specific gravity of gold being 19·2 and that of lead being 11·2, find the ratio which the quantity of matter in 5 cubic inches of lead bears to the quantity of matter in 7 cubic inches of gold.

3. In a certain state of the atmosphere 100 cubic inches of air weigh 31 grains; at a temperature the same as that of the air 30 cubic inches of mercury weigh 14·88 lbs.: find the number of cubic inches of air which contain as much matter as a cubic inch of mercury.

4. If 100 cubic inches of oxygen (under certain circumstances of pressure and temperature) weigh 35 grains, and 1 cubic inch of mercury weighs ·49 lbs., how many cubic inches of oxygen would contain the same quantity of matter as 1 cubic inch of mercury?

5. An iceberg floats with $\frac{9}{10}$ of its volume immersed: find the specific gravity of ice.

6. Define Mass and Density. Of two bodies one has a volume of 5 cubic inches, the other of one-fifth of a cubic foot. In a balance the former weighs 15 oz., the latter 12·8 lbs.: what is the ratio of the mass of the first to that of the second, and what is the ratio of the density of the first to that of the second?

7. A rod of uniform cross section, 18 inches long, weighs 3 oz.; its specific gravity is 8·8: what fraction of a sq. in. is the area of its cross section? The weight of a cubic inch of water may be taken to equal 252 grains.

8. If 8 cubic inches of a liquid whose specific gravity is 1·25 are mixed with 12 cubic inches of another liquid whose specific gravity is 1·125, what is the specific gravity of the mixture if neither shrinking nor expansion occurs?

9. A piece of glass weighs 47 grams in air, 22 grams in water, and 25·8 grams in alcohol: find the specific gravity of the alcohol.

10. A specific gravity bottle full of water weighs 44 grams, and when some pieces of iron weighing 10 grams in air are introduced into the bottle, and the bottle again filled up with water, the combined weight is 52·7 grams: what is the specific gravity of the iron?

11. Define density. The specific gravity of brass referred to water is 8·2. Taking the weight of one cubic foot of water as 1,000 oz., find the density of brass in pounds to the cubic inch

12. If a cubic centimetre of water contained exactly a gram of matter, what would be the quantity of matter—estimated in grams—in a cubic foot of lead, a linear foot being 30·45 centimetres, and the specific gravity of lead being 11·445.

13. Define clearly what is meant by Specific Gravity. Is there any difference in specific gravity between 4 lbs. of iron and 2 lbs. of the same metal?

14. A solid weighs *in vacuo* 100 grains, in water 85 grains, and in another fluid 88 grains: what is the specific gravity of this fluid?

15. A solid, soluble in water but not in alcohol, weighs 346 grains in air and 210 in alcohol: find the specific gravity of the solid, that of alcohol being 0·85.

16. A glass ball weighs 1000 grains; it weighs 630 grains in water and 650 grains in wine: find the specific gravity of the wine.

17. The glycerine barometer is found to stand at 329·2 inches when the mercurial barometer stands at 30·61; given that the specific gravity of mercury is 13·569, find the specific gravity of glycerine.

18. The specific gravity of a substance is 7·5; a portion of it weighs 390 grains in water: what is its weight in air?

Summary.

Measurement of area is obtained directly from the measurement of length.

The unit of area is the area of a square, the edge of which is the unit length. As the unit of length in the British system is the yard, the *British unit of area is the* **square yard.** The smaller units—the *square foot* and the *square inch*—are, however, much more convenient, and are generally used.

The *Metric unit of area is the square metre*, the **square centimetre**, or the area of a square, the length of its edge being one centimetre, is generally used.

Prefixes **deci-**, **centi-**, and **milli-** are used to denote sub-multiples of the unit, and the prefixes **Deka-**, **Hekto-**, and **Kilo-** are used for multiples of the unit.

Unit of volume in the British system is the **gallon**, defined as the volume occupied by 10 lbs. of pure water at a temperature 62° F. A larger unit is the **cubic foot.** A cubic foot is taken to weigh 62·3 lbs., or approximately $6\frac{1}{4}$ gallons, equal to $62\frac{1}{2}$ lbs., or 1000 ounces.

A *Pint of pure water* weighs about $1\frac{1}{4}$ lbs. The sub-multiple (the *cubic inch*) is often used; a cubic inch of water weighs

$$\frac{62{\cdot}3 \times 7000}{1728} = 252\tfrac{1}{2} \text{ grains nearly.}$$

A *cubic yard* contains 27 cubic feet.

The metric unit of volume is the **litre**, or the space occupied by a kilogram of water at 4° C.; a more convenient unit for many purposes is the **cubic centimetre** (c.c.). The mass of a c.c. of pure water at 4° C. is **one gram.**

The **mass of a body** denotes the quantity of matter it contains.

Unit of mass is one pound (1 lb.) defined as the quantity of matter in a platinum cylinder deposited in the Standards Office.

Multiples of the unit are 1 cwt. = 112 lbs., and 1 ton = 20 cwts., or 2240 lbs.

Sub-multiples are obtained by dividing one pound into 16 equal parts, each an *ounce*, or into 7000 parts called *grains*.

The metric unit of mass is the **Kilogram**. This is defined to be the mass of a platinum cylinder deposited in the French archives.

The kilogram is very nearly 1 litre, or 1000 c.c. A more convenient unit is the *mass of* 1 *c.c. of water*, or 1 **gram**.

Weight.—The weight of a body is the attractive force which the earth exerts on a body at or near its surface. The *weight* of a body denotes a *quantity of force*. The *mass* of a body denotes a *quantity of matter*.

Density of a body is the mass of unit volume. **Relative density** of a body is the ratio of its mass to that of an equal volume of a standard substance. The standard substance is pure water, and relative density is usually known as **specific gravity**.

Principle of Archimedes.—A body immersed in water loses weight equal to the weight of the liquid displaced. A floating body displaces a volume of water equal to its own weight.

CHAPTER III.

TIME AND ITS MEASUREMENT. ANGULAR MEASUREMENT; DEGREES, RADIANS. GEOMETRICAL CONSTRUCTION.

Intervals of Time.—A measure of time may be derived from the apparent motion of the sun which appears to an observer to move through certain periodic changes of position; thus, it rises and travels higher and higher until it reaches its highest point, then descends lower and lower until it finally sets. It is difficult, if not altogether impossible, to observe the instant at which dawn commences or darkness begins. It is, however, easy to tell when the sun is at its highest point in the heavens, for the shadow cast by any upright object on the earth's surface would shorten as the sun in its apparent journey travelled higher and higher; at noon, when the sun reaches its highest position, the shadow would have its shortest length, and as the apparent journey continues, the shadow again lengthens.

An **Apparent Solar Day** is the interval of time reckoned from one noon to the next; true noon is the instant when the sun rises to its highest altitude, or, as it is often expressed, when the sun "souths." These apparent days are variable in length, being shortest in November, and longest in February.

Mean Solar Day.—The apparent solar day marks out unequal portions of time; a mean or average solar day is obtained by adding together the lengths of all the days in the year and dividing the sum by the number of days; this gives a mean or average interval of the same duration in each case.

As already indicated, the interval between two consecutive southings of the sun varies throughout the year, but this is not the case with the southing or transit of a star; it is found that the time between two successive transits of a star is invariably the same.

Rotation of the Earth.—The apparent rotations of the stars across the sky are produced by the rotation of the earth, and the interval between any two consecutive returns of a star to the same point of the sky is the exact time taken for one rotation of the earth on its own axis, and is called a sidereal day. On this uniformity our standards of time are made to depend. The result is the same whatever star is selected, thus showing that any and every part of the earth's surface moves with the same angular velocity.

The invariable interval of time between two consecutive southings of a star is divided into 24 equal parts, each called an **hour.** The hour is further subdivided into 60 minutes, and each minute into 60 seconds. Fractional parts of a second are expressed decimally.

A **Mean Solar Year** consists of 366·242 *sidereal days*, or 365·242 mean *solar days.*

Measurement of Time.—The measurement of time by means of clocks and chronometers can be made with great accuracy and precision. These are checked by astronomical observations, especially by means of the transits of stars. By means of one or more good clocks carefully made to agree with astronomical observations, any number of others can, by means of electrical appliances, be made to record accurately the same time as the standard clocks.

Unit of Time.—The unit of time chiefly used is the second. Engineers and others, in the estimation of power, etc., use the larger unit, the minute.

Angular Measurement.—In angular measurement, as in linear measure, a suitable *unit* of measurement is selected, and the number of times that any given angle contains the unit is the numerical measure of the angle.

The two units in general use are the **degree** and the **radian.** The degree is obtained by drawing a circle of any convenient radius, and dividing its circumference into 360 equal parts. If two consecutive divisions be joined to the centre, the two lines so drawn contain a length of arc equal to $\frac{1}{360}$th part of the circumference of the circle, and the angle between them is known as an angle of **one degree.** If, in the circle, two radii are drawn perpendicular to each other, they enclose a quarter of the circle, and hence a right angle consists of ninety degrees, written 90°. Each degree is divided into 60 equal parts, or **minutes**; and each minute is again subdivided into 60 equal parts called **seconds.**

Abbreviations are used for these denominations. Thus, $52° \; 14' \; 20{\cdot}5''$ denotes 52 degrees 14 minutes 20·5 seconds.

The following definitions will often be required.

A **triangle** is a plane figure bounded by three straight lines; any one of the three angular points *A*, *B*, or *C*, Fig. 10, may be looked upon as the **vertex**, the opposite side is then called the **base** of the triangle. The **altitude** of a triangle is the perpendicular distance of the vertex from the base.

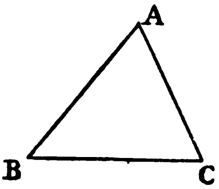


FIG. 10.—A triangle.

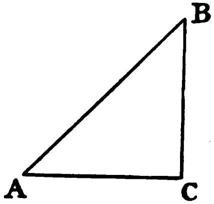


FIG. 11.—Right-angled triangle

Equilateral triangle.—When the three sides of a triangle are equal, the triangle is an equilateral triangle, also the angles of the triangle are equal, each being 60°.

Isosceles triangle.—When two sides of a triangle are equal, the triangle is an isosceles triangle.

A **right-angled triangle,** Fig. 11, is a triangle one angle *C* of which is a right angle; the side *AB* opposite the right angle is called the **hypotenuse.**

A **plane angle** is the inclination of two lines which meet each other but are not in the same straight line. Thus, if a line *AO* meet a line *OC* at *O*, Fig. 12, the amount of opening between the lines *CO*, *OA* is called the angle *AOC*. If only one angle is formed at *O*, the angle may be written as the angle *O*;

but if several angles come together at the same point, the middle letter indicates the *vertex* of the angle referred to. The magnitude of the angle in degrees, minutes and seconds may be indicated as shown in the angle *COA* Fig. 12, or Greek letters may be used for the same purpose. The angle *COA* may be designated by θ, the angle *AOD* by ϕ, and the angle *DOE* by α. It must be very carefully observed that the *angle is independent of the length of the lines forming the sides or legs of the angle.* Thus the angle θ may be accurately described either as the angle *AOC*, or *MON*.

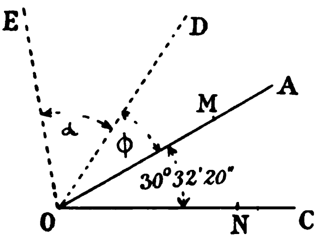


FIG. 12.—Plane angles.

Representation and Measurement of Angles.—As the length of the lines forming the two sides of an angle have no connection with the magnitude of the angle, the actual size is best expressed by the fraction of a circle which the angle in question subtends at its centre. This is done in the following manner.

With centre *O* and any convenient radius, describe a circle *CBDE*, as shown in Fig. 13.

If we suppose a small pointer (such as the minute hand of a clock or watch) free to move about the centre *O*, made to coincide with *OC* and afterwards made to move from *C* towards *B* to a position *A*, through an arc *CA* one-sixth of the circumference, then the angle *COA* is a sixth part of 360°, or is an angle of 60 degrees, written as 60°. In a similar manner, if it moves to *B* it will trace out an angle of 90 degrees.

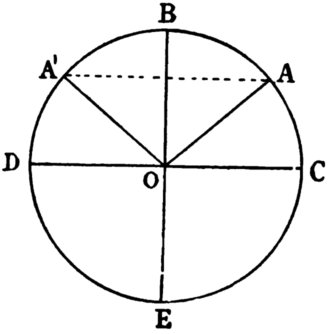


FIG. 13.—Measurement of angles.

When it moves to a position A' it is evident that the angle traced out is greater than a right angle. All angles greater than a right angle are called *obtuse* angles. Consequently, the angle COA' is an obtuse angle. Angles less than 90°, or less than a right angle, are *acute*. The angle COA is an acute angle. From the foregoing considerations, it will be seen *that an angle is measured by the number of degrees in the arc of the circle, having the vertex of the angle as its centre, intercepted by the two lines forming the angle.*

Comparison of the Magnitudes of Angles.—A comparison of the magnitudes of two angles ABC and DEF (Fig. 14) may be made by placing the angle DEF on the angle ABC, so that the point E exactly falls upon the point B, and the line DE along the line AB, then if the line EF falls on the line BC the angles are said to be equal. The angle DEF is less than the angle ABC if EF falls within BC, as shown by the dotted line BC'. It is larger if it falls outside BC. This method of *superposition* is readily performed in the following way: Draw from centres B and E, as shown, two equal arcs AC and DF, so that DE and EF in the one case are equal to AB and BC respectively in the other. If the point A be joined to the point C, and D to F, then, if AC is equal to DF, the triangles ABC and DEF are obviously equal; or, using a piece of tracing paper, make a tracing of DEF, and placing the tracing on ABC, the comparison is readily made.

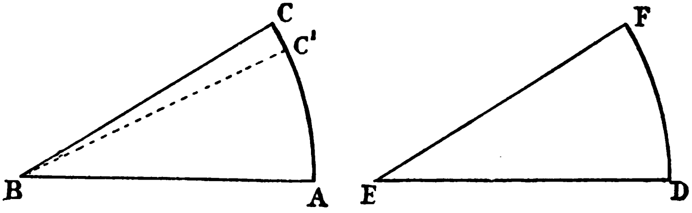


FIG. 14.—Comparison of the magnitudes of two angles.

To set out a given angle.—By means of what is called a 60° set square shown at (i) (Fig. 15), angles of 60°, 30°, 90° and 120° can be set out. Also by means of the 45° set square shown at (ii), angles of 45°, 90°, and 135° can be marked off. Other angles, viz. 15°, 22½°, and 75° can also be obtained by using these set squares and a pair of compasses.

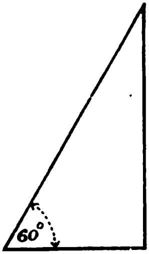


FIG. 15 (i).—A 60° set-square.

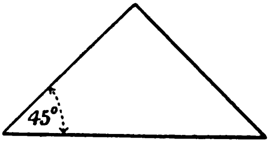


FIG. 15 (ii).—A 45° set-square.

To set out an angle of 15°.—Make an angle *ABC* (Fig. 16) equal to 30° with the set square (i). Bisect the angle *ABC* by a line *BD*, then *ABD* and *DBC* are each 15°. To bisect the angle *ABC* we proceed as follows:—With *B* as centre, and any convenient radius *BA*, describe an arc *AC*. With *A* as centre and any radius describe an arc, and from *C*, using the same radius, describe another arc cutting the former in point *D*. Join *D* to *B*, then *BD* bisects the angle *ABC*, and therefore makes an angle of 15° with *AB*.

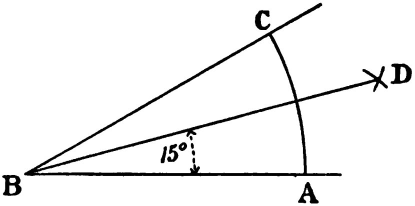


FIG 16.—To set out an angle of 15°.

An angle of 22½° is obtained in a similar manner. Make an angle *ABC* (Fig. 17) equal to 45° with the set square (ii). Bisect the angle *ABC* by the line *BD* as described above; then the angle *ABD* is equal to 22½°.

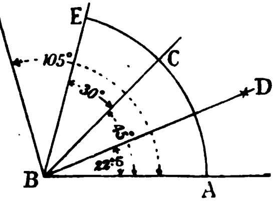


FIG. 17.—To set out an angle of 22½°.

An angle of 75°.—To obtain an angle of 75° it is only necessary to use the two set squares. Thus the angle ABC is made equal to 45° by using set square (ii), next adding to this the angle 30° by means of set square (i). The angle $ABE = 75°$.

If the 60° angle of set square (i) be added to the angle ABC (Fig. 17) the angle obtained will be 105°.

Use of Protractor.—Angles which are not conveniently obtained by construction are set out by means of a **protractor.** Two forms of such protractors are shown in Fig. 18. The first consists of a thin flat rule or scale made of boxwood, ivory, or other material, along the edge of which angles are marked. These marks are obtained from the corresponding division on a semicircle as shown in the illustration.

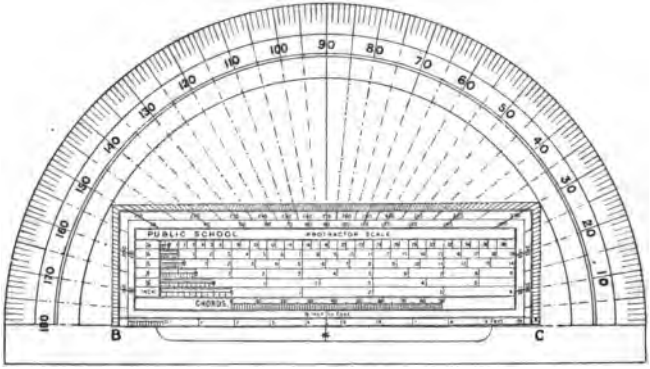


Fig. 18.—Two forms of protractors.

Ex. 1. Set out by a protractor an angle of 50°.

At the point P (Fig. 19) we place the mark * of Fig. 18 coincident with P, and the edge of the protractor BC with the line PM.

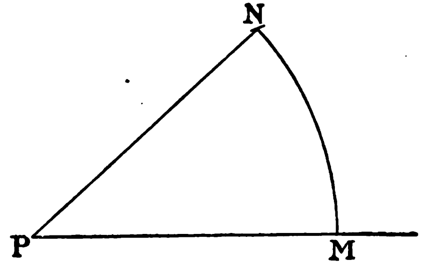


Fig. 19.—To set out an angle by means of a protractor.

Make a mark opposite the division indicating 50° on the protractor. Remove the protractor and join the mark to *P*. An angle *MPN* containing 50° will have been made with the given line *PM*.

How to Use a Protractor to Measure an Angle.—In a similar manner, when used to measure a given angle, the edge *BC* of the protractor is placed coincident with one of the lines forming the angle, the mark * on the protractor coinciding with the vertex of the angle, the point where the other line crosses the divisions on the scale would be noted; this would be in degrees the angle required.

Scale of Chords.—At the lower part of a protractor there is nearly always a scale, as shown in Fig. 18, marked "CHORDS." It is known as a scale of chords, and this scale can be used to set out an angle. To do this proceed as follows:—set a pair of dividers to the space between 0° and 60° on the scale of chords, and, with this distance as radius and *P* as centre, describe an arc of a circle large enough to contain the required angle. Again apply the dividers to scale of chords and measure a space equal to the number of degrees required. Then, with *P* as centre and the length so obtained as radius, cut the arc *MN* at *N*. Join *N* to *P*. Then *MPN* is the angle required (Fig. 19).

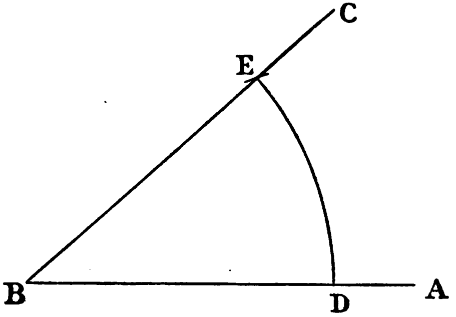


FIG. 20.—To measure the magnitude of an angle.

Similarly to measure any given angle *ABC* (Fig. 20) by means of a scale of chords. Using the compasses obtain a distance 0° to 60°, and with this distance as radius make an arc *DE*.

Place one point of the compasses on D and the other coinciding with E. Apply the measurement DE to the scale of chords. The reading will give the magnitude of the angle ABC.

Measurement of an Angle in Radians.—A very convenient method, which is especially useful when dealing with angular velocity, is obtained by measuring or estimating the arc subtended by a given angle, and dividing the length of the arc by the radius of the circle.

Thus, in Fig. 21, the measure of the angle BOA in radians is the length of arc AB divided by the radius OA, or

$$\text{Measure of angle in radians} = \frac{\text{length of arc } AB}{\text{length of radius } OA}.$$

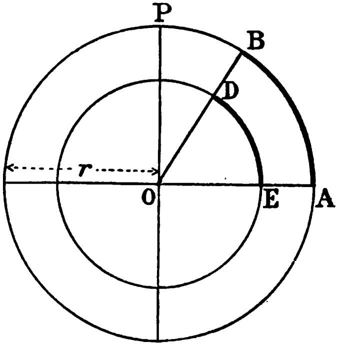


FIG. 21.—Measurement of an angle in radians.

Evidently this measure of an angle will be unity when the arc is equal to the radius, or the *unit angle is that angle at the centre of a circle subtended by an arc equal in length to the radius.* This unit is called a **radian.**

To obtain the value of the unit angle in degrees you may argue as follows: the circle contains 4 right angles or 360°. This may be expressed in radians by the ratio of

$$\frac{\text{circumference of circle}}{\text{radius of circle}},$$

but the circumference of the circle as shown in p. 241 is $2\pi r$. Where r is the radius OA and π denotes the number of times that the diameter of a circle is contained in the circumference, the value of π is 3·1416, or more accurately 3·14159.

Hence the measure of 4 right angles $= \frac{2\pi r}{r} = 2\pi$. Thus 4 right angles may be expressed in two ways, viz., as 360°, or as 2π radians, and one radian $= \frac{360}{2\pi} = \frac{180}{\pi} = 57^\circ{\cdot}29578$ (taking $\pi = 3{\cdot}1416$). For many purposes the approximate value $57^\circ{\cdot}3$ is used instead of the more accurate value.

The angle as you have previously learnt, formed by two lines, does not depend on the length of the lines but upon their relative directions. Now it will be seen from Fig. 21 that if any circle be drawn with centre O and cutting the lines OA and OB in any two points such as D and E, then the angle $EOD =$ angle AOB. It follows, therefore, that the unit angle is independent of the size of the circle and is an invariable unit being, as already indicated, equal to $\frac{180}{\pi}$, or $57^\circ{\cdot}2958$.

Circular Measure of an Angle.—The older method of the measurement of an angle in radians is referred to as the circular measure of an angle, but it is much better for all practical purposes to use the term radian measure instead of the now obsolete term circular measure.

It is advisable not only to be able to define the two units, the *degree* and the *radian*, but also to realise their relative magnitudes. Thus an angle 5° denotes an angle of 5 *degrees*, but an angle simply denoted by 5 contains $5 \times \frac{180}{\pi}$ degrees.

Also when π refers to a number it denotes 3·1416, but applied to an angle, then the angle contains π radians and is 180 degrees.

Ex. 1. An angle is $\frac{2}{3}$ radians, what is its value in degrees?

In this example, as the angle is $\frac{2}{3}$ in circular measure, its value in degrees will be $\frac{2}{3} \times$ unit angle, or $\frac{2}{3} \times 57^\circ{\cdot}2958 = 38^\circ{\cdot}1972$.

Ex. 2. (*a*) What is the numerical value of a right angle in radians? (*b*) Find the radian measure of an angle of $112^\circ\ 43'$. (*c*) Find the length of an arc which subtends an angle of $112^\circ\ 43'$ at the centre of a circle whose radius is 153 feet.

(*a*) The measure of 4 right angles is 2π radians, therefore the measure of 1 right angle is

$$\frac{2\pi}{4} \text{ radians} = \frac{\pi}{2} \text{ radians} = 1{\cdot}5708 \text{ radians.}$$

(*b*) In each degree there are 60 minutes, hence $112^\circ\ 43' = 6763$ minutes. Also $180^\circ \times 60 = 10800$ minutes;

$$\therefore \frac{6763 \times 3{\cdot}1416}{10800} = 1{\cdot}967 \text{ radians.}$$

(*c*) In radian measure, $\text{angle} = \frac{\text{length of arc}}{\text{radius}}$;

$$\therefore \text{ length of arc} = \text{angle} \times \text{radius} = 1{\cdot}967 \times 153$$
$$= 301 \text{ feet nearly.}$$

EXERCISES. II.

1. What are the numerical values of a right angle in radian measure, and in degrees?

Find in radian measure the numerical value of the angle 37° 15′.

2. Define the radian or unit angle, and write down in a decimal form, true to five places, the value of the ratio of the circumference of a circle to its diameter.

Find, as accurately as you can, the numerical value in radian measure of 45″ and of 25° 10′.

3. What is the numerical value of two right angles in radians?

If one of the acute angles of a right-angled triangle is 1·2 radians, what is the numerical value of the other acute angle (*a*) in radians, (*b*) degrees, minutes, etc.?

4. Find the number of degrees in an angle whose radian measure is $\frac{1}{8}$.

5. How many radians are there in the angle through which the minute hand of a clock revolves between 5 minutes after 9 o'clock and 5 minutes after 12 o'clock?

6. In the last question if the outer diameter of the graduated dial is 6 feet, find the distance between any two consecutive minute divisions.

7. Define a radian, and taking $\pi = 3{\cdot}1416$ calculate the number of radians in an angle of 120°.

8. Find the diameter of a graduated circle if the value of each of the divisions on the outer rim is five minutes, and the distance between successive graduations is 0·1 inch.

9. Explain the two methods of measuring angles that are chiefly used.

Find the length of an arc on the sea which subtends an angle of one minute at the centre of the earth, supposing the earth a sphere of diameter 7920 miles.

10. Show by means of a figure what you understand by an angle of one radian. How many degrees are there in it?

A man walks a quarter of a mile round a circular track, and it is found that this portion subtends at the centre an angle of $2\frac{1}{5}$ radians: how many times must he go round before he walks 10 miles, and what is the radius of the track?

11. Find the measure in radians of the following angles: 10°, 20°, 30°, 50°, 90°, 180°. Verify by Table VI.

12. How many radians are there in the angle through which the minute hand of a watch revolves between 5 minutes after 10 o'clock and 3 minutes after 12 o'clock?

13. What is the measure of an angle of 80° in radians?

Express in degrees, minutes, and seconds the angle in a segment of a circle whose arc is $\frac{1}{7}$th of the circumference of the circle,

14. Show the truth of the equations $\frac{d}{180}=\frac{\theta}{\pi}$, where d and θ are respectively the number of degrees and radians in an angle.

Find the number of degrees in an angle whose measure is $1\frac{1}{9}$ radians.

15. Explain the different methods of measuring angles.

Find which is greater, an angle of 66° or an angle whose measure is 1·15 radians.

16. Find which is greater, an angle of 132° or an angle whose measure is 2·3 radians.

17. Define a degree, and assuming $\pi=\frac{22}{7}$, find the number of degrees in the angle whose radian measure is ·1.

18. Taking $\pi=3\cdot1416$, find correct to three places of decimals how many degrees there are in a radian.

One end of a string 4 feet long is fastened to a point C, and at the other end is attached a small weight which is made to swing in a vertical plane, so that in its highest position the weight is 2 feet below the level of C. Find the number of degrees and also the number of radians in the complete angle of swing.

19. One angle of a triangle is 45° and another is $\frac{5\pi}{8}$ radians; express the third angle both in degrees and in radians.

20. If a wheel be 5 feet in diameter, and a point on its circumference has a speed of 10 feet per second, express in radians the angle turned through in $\frac{1}{4}$ second.

How many revolutions will the wheel make per minute?

21. What is π? What is 'the angle whose circular measure is π'?

If an arc of 12 feet subtend at the centre of a circle an angle of 50°, what is the radius of the circle?

22. (i) Express an angle of 36° 15′ 12″ as the decimal of a radian correct to five places.

(ii) The distance between successive graduations on the outer rim of a graduated circle is one-fiftieth of an inch, and this interval subtends an angle of one minute at the centre of the circle: find the diameter of the circle.

(iii) If the radius of a circle is 10,000 ft., find to the nearest hundredth of a foot the length of the arc, which subtends an angle of one minute at the centre.

Geometrical truths demonstrated by use of instruments.—The following examples are to be proved by construction, *i.e.* by use of drawing instruments, *compasses*, *set-squares*, a good *scale*, and a *protractor*.

The three angles of a triangle are together equal to 180°.—Draw the triangle ABC (Fig. 22) with the angle at C a right angle. Make the two sides a and b respectively equal to 4 and 3 units of length. Join AB, measure by the protractor the angles at A and B (p. 29); add the values of the angles measured in degrees together, and ascertain if the angles A, B, and C make 180°. It will be found that $A=53° \; 7'$ and $B=36° \; 53'$.

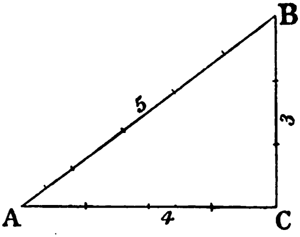


Fig. 22.—The three angles of a triangle are together equal to 180°.

The greater angle of every triangle is subtended by the greater side.—Draw a triangle having its sides 9, 7, and 3 units, measure the angles with a protractor, and carefully observe that the greatest angle is opposite the greatest side 9, and the least angle is opposite the smallest side 3.

EXERCISES. III.

1. If the three sides of a triangle are 40, 30, and 50 units of length respectively, find the three angles of the triangle measured in degrees. Draw a triangle to scale, and show by construction that the length of the third side must be less than the sum of the lengths of the other two sides, and hence satisfy yourself that any *two sides of a triangle are together greater than the third side.*

2. Given the three sides of a triangle $a=19{\cdot}7$, $b=5{\cdot}3$, $c=24$, find the angles.

3. The sides of a triangle a, b, c are proportional to 4, 5, and 6: find the angle B

4. If the hypothenuse AB of a right-angled triangle $=272$ ft., and the perpendicular $BC=232$ ft., find by measurement the length of the side AC. By means of the protractor find the angle at A. Also measure the angle at B. See if the sum of the three angles of the triangle are equal to 180° by adding the observed values, also by adding the angles together graphically by means of a piece of tracing paper.

5. Given the base and perpendicular of a right-angled triangle $=186$ and 152 respectively, find by measurement the hypothenuse and the angle at the base.

6. In a triangle one side a is 266, the angle A is 38°, the angle B is 72°: find the other two sides by construction.

7. In a triangle ABC the side AB is 345, the side AC is 174, and the angle A is 37°: find the remaining side and the other angles.

8. Draw a triangle having its sides respectively 6, 7, and 9 units in length. Measure the angles with a protractor.

The angles at the base of an isosceles triangle are equal.—Draw to scale an isosceles triangle ABC, the side AB = side AC (Fig. 23). Show by measuring the angles that the angle at C is equal to the angle at B.

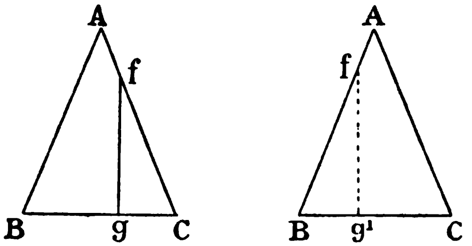


FIG. 23.—To show that the angles at the base of an isosceles triangle are equal.

Again, prove the equality by cutting out the angle at C and placing it on B. This may be effected by marking off a length Cg not greater than half BC, and drawing gf perpendicular to BC meeting AC in f. If Cfg be placed as shown in Fig. 23, the angle C on B and Cg coinciding with Bg, then the line Cf will be found to coincide with the line BA.

Or, using a piece of tracing paper, trace the triangle, turn the paper over and see that the angles are equal.

If a line be drawn at right angles to the base, and passing through the vertex, it will bisect the base. Draw a triangle ABC and AD at right angles to the base. Make a tracing of the triangle ACD, then if placed on the triangle ABD, with the point C coincident with B, all the other lines of the triangles can be made to coincide. Hence the triangles ACD and ABD are equal, and D is the middle point of BC.

Equilateral triangle.—Make a triangle having all three sides equal. (*a*) Measure any one of the three angles and write down its magnitude; (*b*) carefully trace one of the angles on a piece of tracing paper, and placing the paper on each of the other two angles, see that all the angles are equal; hence the sum of the three is 180°.

Parallel Lines.—*When two parallel straight lines are crossed by a third straight line, the alternate angles are equal.*

Draw any two parallel lines (Fig. 24), and a third line *EF* crossing them. Show, by tracing the angles on a sheet of paper, or by measuring the angles, that the alternate angles marked × are equal to each other, and similarly those marked *O* are equal. Also show by measurement that the four angles formed by the intersection of the third line are equal to 360°.

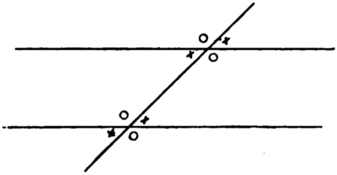


FIG. 24.—Parallel lines.

If a triangle and a parallelogram are on the same or equal bases, and the altitude of each is the same, the area of the parallelogram is double that of the triangle.—Draw any parallelogram *ABCD*; join *A* to *C* (Fig. 25). By cutting the paper along *AC* the triangle *ABC* can be made to coincide with the triangle *ADC*; or, using a piece of tracing paper, trace carefully the triangle *ABC*, then placing it on *ADC* with *B* at *D*, note that the lines forming the triangle are coincident.

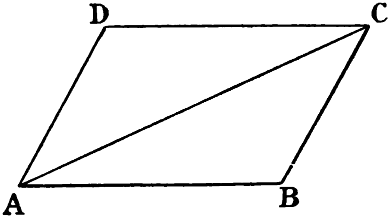


FIG. 25.—If a triangle and a parallelogram are on the same base and of equal altitude, the area of the parallelogram is double that of the triangle.

In any circle the angle at the centre is double the angle at the circumference, on the same, or on equal arcs as bases.

Draw a circle of 3 or 4 inches diameter, select any two points, *A* and *C* (Fig. 26), on the circumference and join to centre *O*. Also join *A* and *C* to any point *B* on the circumference.

Measure the angles AOC and ABC by the protractor and prove the theorem.

Make $AD=AC$; join A and D to any point E on the circumference; show that the angle

$$ADE=ABC=\tfrac{1}{2}AOC.$$

Angles on the same, or on equal arcs, are equal to one another. Prove this by joining points DA, and AC to different points on the circumference.

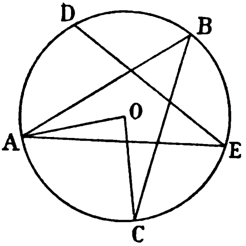


FIG. 26.—On the same or equal arcs the angle at the centre is double the angle at the circumference.

If a line be drawn parallel to the base of a triangle, cutting the side or sides produced, the segments of the sides are proportional.

Draw a triangle ABC, and a line DE parallel to the base (Fig. 27); show by measurement that

$$\frac{BC}{AB}=\frac{DE}{AD};$$

∴ if DE is one-half BC, then AD is one-half of AB.

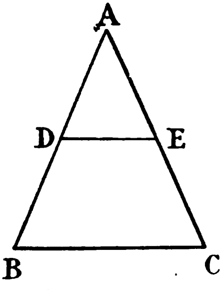


FIG. 27.—Similar triangles.

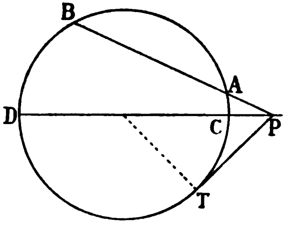


FIG. 28.—$PB\times PA=PT^2$.

If a tangent PT be drawn to a circle from a point P (Fig. 28), the rectangle contained by $PB\,.\,PA$ or by $PD\,.\,PC$ is equal to the square on the tangent PT;

$$\therefore\quad PD\times PC=PT^2,\ \text{or}\ 2r\times PC=PT^2.$$

If two chords in a circle cut one another, the rectangle on the segments of one of them is equal to the rectangle on the segments of the other.

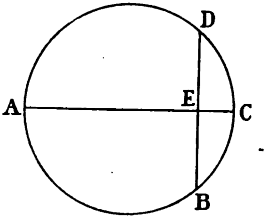


FIG. 29.—$AB \times EC = BD^2$.

Thus, if AC and BD be two chords in a circle cutting each other at a point E, the rectangle $AE \, . \, EC =$ rectangle $BE \, . \, ED$.

If, as shown in Fig. 29, the lines are perpendicular to each other, and one passes through the centre, then $BE = ED$;

$$\therefore \; AE \, . \, EC = ED^2.$$

From this the graphic method of obtaining square root as shown on p. 137 is obtained.

Summary.

Apparent Solar Day is the interval of time reckoned from one noon to the next.

Mean Solar Day.—By adding the lengths of all the days in one year together and dividing by the number of days *we obtain* what is termed a *mean solar day.* This is divided into 24 equal parts, each an *hour*; each hour is divided into 60 minutes and each minute into 60 seconds.

Rotation of the Earth.—The interval of time between two successive transits of a star is always the same, and is the time taken for one rotation of the earth upon its axis.

Sidereal Day.—The time taken for one rotation of the earth on its axis is invariably the same, and is called a *sidereal day.* On this uniformity our standards of time are made to depend.

Mean Solar Year consists of 366·242 *sidereal* days, or 365·242 mean *solar* days.

Measurement of Time is made by clocks and watches.

The standards are checked by astronomical observations.

The unit of time, adopted by practical men and used in all scientific work, is the **second.** The larger unit, 1 minute, is often used by engineers in the estimation of power.

In angular measurement the *units* used are the **degree** and the **radian.** The magnitude of the latter is nearly 60 times that of the former. 1 radian = 57·3 degrees.

CHAPTER IV.

ARITHMETIC. SIGNS OF OPERATION: "× AND ÷"; G.C.M.; L.C.M.; VULGAR FRACTIONS; DECIMALS.

THE symbols or signs of operations used in Arithmetic and Algebra are very convenient, and it is advisable to become familiar with them as soon as possible. It is hoped that the majority of the symbols which will be introduced as required in the following pages are already known; if this is not so, they should be used on all possible occasions until they can be used quickly and accurately.

Signs of Operation: Addition.—The sign + (called plus) indicates that the number before which it is placed is to be added to the number in front of it. Thus $4+5$ means that 5 is to be added to the number 4; also $4+5+7$ means that 7 is to be added to 5, and then 4 to the result, or, simply the three numbers are to be added together.

The sign = is used as an abbreviation of the words "is equal to," or "equals." Thus $4+5+7=16$ is read as 4 plus 5 plus 7 equals 16.

In like manner $a+b$ indicates that a quantity represented by b is to be added to a quantity represented by a, giving as a result $a+b$.

Subtraction.—The sign − (called 'minus') is the sign of Subtraction, and indicates that the number which follows it is to be taken away from the number which is placed before it.

Thus $12-4$ (read as 12 minus 4) means that 4 is to be subtracted from 12, hence $12-4=8$. In a similar manner $a-b$ indicates that a quantity b is to be subtracted from a quantity a.

Multiplication.—When one number is multiplied by another the sign × is placed between the numbers; thus 6×4 means "6 multiplied by 4," or 6 times 4.

From Fig. 30, where 4 rows are arranged, each row containing 6, the result of multiplying 6 by 4 can be obtained by the simple process of addition; although this method could be used when the numbers are small, it would be very troublesome with large numbers.

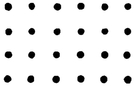


FIG. 30.—To show that $6 \times 4 = 4 \times 6$.

Also from Fig. 30, as there are 6 vertical rows each containing 4, the product is 6×4; there are 4 horizontal rows each containing 6, and the product is 4×6; hence $4 \times 6 = 6 \times 4$.

The number 6 preceding the × is called the **multiplicand**; the number 4 following it is called the **multiplier**, and the final result is called the **product**. Also, as $6 \times 4 = 4 \times 6$, the multiplicand and multiplier are interchangeable. The two **numbers** themselves are called the **factors** of the number denoting the **product**. The product of the two numbers may be shown graphically by drawing a line AB equal to 6 units (Fig. 31), and a line AD at right angles to AB, and equal to 4 units; by drawing lines through the points of division parallel to AB and AD as shown, the product is seen to consist of 24 equal squares.

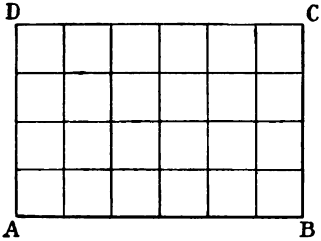


FIG. 31.—Product of two numbers.

If the two numbers were denoted by a and b, the product would be $a \times b$, or, as usually written, ab.

Division may be indicated by placing the symbol ÷ between two or more quantities; thus $8 \div 2 = 4$ (which is read as 8 divided by 2 is equal to 4, or 2 into 8 is 4); the division is often expressed by writing the two numbers in a fractional form, as $\frac{8}{2} = 4$.

In a similar manner, to divide a quantity b by a quantity a, we write it as $\frac{b}{a}$.

The sign of equality (=) indicates that the quantities between which it is placed are equal to one another, or have the same numerical value. Thus $\frac{6}{3}=2$.

The number preceding the ÷ is called the **dividend,** the number following it is called the **divisor,** and the result, when the operation indicated by the sign is performed, is called the **quotient.**

It will be seen from the above that the signs or symbols of operations give definite information, enabling operations and results to be written in a brief and clear manner.

The operations just described may be summarised by simple examples as follows:

$$4+5=9; \qquad 12-4=8;$$
$$5\times 6=30; \quad 8\div 2, \text{ or } \tfrac{8}{2}=4.$$

Prime number.—Any number which is divisible by no other number except unity is called a prime number; thus 1, 3, 5, etc., are prime numbers.

Measure, multiple, or factor.—A number which is contained an exact number of times in another number, is said to be a measure or factor of the latter; it is also called a sub-multiple of the number.

Greatest Common Measure.—The greatest number which is contained an exact number of times in each of two or more given numbers is called their Greatest Common Measure, or, as usually written, G.C.M.

Thus, to find the G.C.M. of the numbers 45 and 126.

On breaking into factors, we have

$$45=3\times 3\times 5; \quad 126=3\times 3\times 7\times 2;$$

and the G.C.M. is $3\times 3=9$.

Ex. 1. Find the G.C.M. of 21879 and 14872;

$$21879=9\times 11\times 13\times 17; \quad 14872=8\times 11\times 13\times 13.$$

The G.C.M. is $11\times 13=143$.

Hence, one method which may be used to find the G.C.M. of two or more numbers is to break them up into factors.

When, however, the numbers are large, the process of finding the factors is rather tedious, and it is better to use the following:

To find the G.C.M. of two numbers. *Divide the larger by the smaller; divide the smaller by the remainder; divide the first remainder by the new remainder, if any, and so on until there is no remainder. The last divisor is the* G.C.M. *required.* The following Example will show how the G.C.M. is obtained.

Ex. 2. Find the G.C.M. of 21879 and 14872.

```
14872 ) 21879 ( 1
        14872
        ─────
         7007 ) 14872 ( 2
                14014
                ─────
                  858 ) 7007 ( 8
                        6864
                        ────
                         143 ) 858 ( 6
                               858
                               ───
```

∴ G.C.M. = 143.

In finding the G.C.M. of two or more numbers, the work is often simplified by taking out any common factor which can be found before any division is made. This common factor must be included in the G.C.M.

When the G.C.M. of more than two numbers is required, the G.C.M. of two may be found, and afterwards the G.C.M. of the result and the third number, and so on.

EXERCISES. IV.

Find the G.C.M. of

1. 2301 and 3717.
2. 116039, 122067, and 137137.
3. 3024, 4752, and 7488.
4. 13536 and 23148.
5. 3761034 and 1081.
6. 91, 5798, and 36595.
7. 132, 236511, and 37499 ; 14003 and 11591.
8. 16650, 10730, and 1961.
9. 1914 and 26361.
10. 7511 and 41107.
11. 6409 and 7395.
12. 5203 and 3741.
13. 15625 and 2625.
14. 22341 and 256583.
15. 106499 and 256583.
16. 9691 and 459001.
17. 5547 and 147008443.
18. 6281 and 326041.
19. 5775 and 3773.
20. 48849 and 59133.
21. 66150 and 145800.
22. 5544 and 2304.
23. 111540, 42336, and 67392.
24. 10395 and 16819.
25. 52, 78, and 416.
26. 8214, 1110, and 1702.
27. 8996427 and 9078399.

Least Common Multiple.—*When one number contains another an exact number of times the first number is called a multiple of the second.* Thus 15 is a multiple of 3 or 5 : 12 is a multiple of 3, 4, and 6.

The smallest number which contains each of two or more numbers is called the Least Common Multiple of the given numbers and is usually written L.C.M.

The L.C.M. of two numbers can be found by resolving the numbers into their prime factors. Select the highest powers of each prime factor which occurs in these products; by multiplying these highest powers together the L.C.M. is obtained.

Ex. 1. Find the L.C.M. of 42, 56, and 63.

Here
$$42 = 3 \times 7 \times 2;$$
$$56 = 8 \times 7 = 2^3 \times 7;$$
$$63 = 9 \times 7 = 3^2 \times 7.$$

The highest powers of 2, 3, and 7 that occur are 2^3, 3^2, and 7.

Hence the $\text{L.C.M.} = 2^3 \times 3^2 \times 7 = 8 \times 9 \times 7 = 504.$

A very convenient and expeditious method of finding the prime factors is as follows: arrange the given numbers in line, separating them by commas as shown. Divide them by any prime factor which is a factor of two or more of them.

Thus, in the above example, dividing by 2, next by 3, and finally by 7, we obtain in the last line the numbers 1, 4, 3. The L.C.M. is the product of the numbers in the last line and the numbers which have been used as divisors.

$$\begin{array}{r|l} 2 & 42,\ 56,\ 63 \\ \hline 3 & 21,\ 28,\ 63 \\ \hline 7 & 7,\ 28,\ 21 \\ \hline & 1,\ 4,\ 3 \end{array}$$

$$\therefore\ \text{L.C.M.} = 2 \times 3 \times 7 \times 4 \times 3 = 504.$$

The G.C.M. of two numbers contains all the prime numbers and powers of prime numbers common to the two numbers, hence to find the L.C.M. we have the rule: Divide the product of the numbers by their G.C.M. In applying this rule the convenient method is to *divide the* G.C.M. *into one of the numbers and multiply the quotient by the other number.*

Ex. 2. Find the L.C.M. of 2145 and 1287.

In the usual way we find the G.C.M. = 429.

Hence $\text{L.C.M.} = \frac{1287}{429} \times 2145 = 6435.$

It is obvious from the preceding that the product of two numbers = G.C.M. × L.C.M.

An old rule for finding the L.C.M. of three or more numbers may, in some cases, be used with advantage. The rule is: Find the L.C.M. of two of the numbers, find the L.C.M. of the third number and the L.C.M. already obtained, and so on.

Ex. 3. Find the L.C.M. of 42, 56, and 63.

The G.C.M. of 42 and 56 is 14. The L.C.M. is $\frac{42}{14} \times 56 = 168$.

The G.C.M. of 168 and 63 is 21.

$$\therefore \text{ L.C.M. is } \frac{168}{21} \times 63 = 504.$$

Ex. 4. Find the G.C.M. and L.C.M. of 1920 and 756.

$$1920 = 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 3 \times 5$$
$$756 = 2 \times 2 \times 3 \times 3 \times 7.$$

The G.C.M. is the product of the common factors

$$2 \times 2 \times 3 = 12.$$

The L.C.M. $= 1920 \times 3^2 \times 7 = 120960$.

EXERCISES. V.

Find the Least Common Multiples of

1. 15, 20, 24, 30.
2. 24, 11, 16, 33, 48.
3. 44, 48, 52, 96.
4. 192, 204, 272.
5. 165, 198, 270.
6. 3024, 4752, 7488.
7. 14003, 11591.
8. 8, 18, 28, 38, 48.
9. 15, 27, 60, 108, 56.
10. 6, 32, 198, 24, and 112.
11. 7, 14, 21, 28, 35, 63.
12. 6, 8, 11, 16, 20.
13. 876 and 864.
14. 1, 2, 3, 4, 5, 6, 7, 8, 9, 10.
15. 68, 41, and 82.
16. 868 and 854.
17. 72, 12, 3, 7, 48.
18. 36, 5, 48, 9, 54, 8, 7.
19. 52, 12, 4, 13, 5, 26.
20. 4, 7, 18, 8, 25, 9.
21. 25, 75, 125, 150, 180.
22. 14, 85, 105, 77.
23. 3, 17, 9, 51, 34.
24. 16, 24, 18, 49, 105, 30.
25. 153, 204, and 221.
26. 11704 and 101080.
27. 3696, 286, and 19656.
28. 8085 and 4125.
29. 111, 37, 8, 12, 3.

Fractions.—A *fraction* is a part or parts of a whole. Suppose that any unit (such as a rod one yard long) is divided into three equal parts, each of these parts is called one-third ($\frac{1}{3}$) of the whole, two of these parts together make two-thirds ($\frac{2}{3}$). Each of these quantities $\frac{1}{3}$, $\frac{2}{3}$ is called a fraction.

The section of Arithmetic dealing with the subject of fractions is so very important that it should receive the best attention of the student. In (Fig. 32) one end of an ordinary foot-rule ($\frac{1}{3}$ yard) is shown. The distance between *a* and *b* or *b* and *c* (one-twelfth part of a foot, or one inch as we have learnt to call it), may be divided into 8 or 10, and on some scales into 12 equal parts. Such a rule or scale, which every student is supposed to possess, is a useful instrument with which to illustrate the four operations of Addition, Subtraction, Multiplication, and Division.

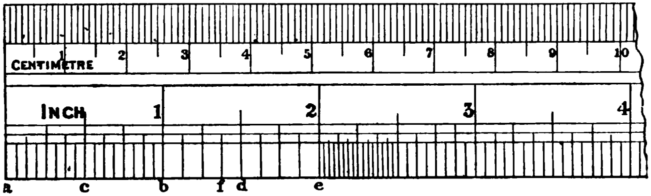


FIG. 32.—Showing one end of a steel scale.

Consider any length on your scale, such as from *b* to *f*, the third division towards *e*. This length is spoken of as three-eighths of an inch, and is written $\frac{3}{8}''$. The number below the line is called the **denominator** of the fraction, and indicates the number of parts into which the unit has been divided. The upper number of the fraction, in this case 3, is called the **numerator** of the fraction, and expresses the number of parts, in this case eighths, which have been taken. Or, generally,

$$\text{Fraction} = \frac{\text{numerator}}{\text{denominator}}.$$

When the numerator and denominator of a fraction are each multiplied or divided by the same number, the value of the fraction is unaltered. Thus, for example, if we multiply both the numerator and the denominator of our fraction $\frac{3}{8}$ by 2, we obtain as a result the fraction $\frac{6}{16}$, and this simply indicates that the unit *ab* (Fig. 32) is divided into sixteen parts instead of eight. Each part is therefore twice as small as before. Hence, in order to obtain a fraction equal to the original fraction, twice as many of these smaller parts must be taken, *i.e.* 6 of the smaller divisions (the unit divided into 16 equal parts) are exactly equal to 3 divisions (the unit divided into 8 parts). In a similar manner it follows that if the numerator and denominator are both divided by the same number, the value is unaltered. Thus, taking 4 divisions from *b* to *d* on the scale, we could write this as $\frac{4}{8}$, dividing numerator and denominator by 4, the fraction becomes $\frac{1}{2}$, the first fraction denoting that the unit *be* is divided into 8 parts, and 4 parts from *b* to *d* are taken; the second form of the fraction corresponds to the division of the unit into two equal parts *bd* and *de*, and one of the equal parts is taken, hence the value of the fraction is unaltered.

By such simple methods as those suggested, it is possible to quite easily understand the above propositions.

Addition, Subtraction, and Comparison of Fractions.—To add or subtract fractions having the same denominators, it is only necessary to add or subtract their numerators. When the denominators are not alike, the fractions must first be reduced to equivalent fractions, each having the same denominator.

Ex. 1. Thus, to add $\frac{7}{20}$ and $\frac{6}{20}$ of a pound together, since the 7 and the 6 in the numerators indicate the number of shillings, the answer is 13 shillings or $\frac{13}{20}$ of a pound; by subtracting one from the other we get $\frac{1}{20}$ of a pound.

To add $\frac{7}{8}$ and $\frac{3}{16}$ together, we first make $\frac{7}{8}$ into $\frac{14}{16}$, then adding the numerators of the fractions we obtain $\frac{17}{16}$. This simple example should, if necessary, be verified by the foot-rule.

Proper Fractions.—Such fractions as $\frac{7}{8}$, $\frac{3}{16}$, etc., in which the numerator is less than the denominator, are known as proper fractions.

Improper Fractions.—Those fractions in which the numerator is not less than the denominator are called improper fractions, and it will be seen that any whole number can always be expressed in a fractional form, thus $6=\frac{6}{1}=\frac{12}{2}=\frac{24}{4}$, etc.

Hence, to represent a number such as $6\frac{1}{9}$ in a fractional form, we express 6 as a fraction with denominator 9, thus, $\frac{6}{1}=\frac{54}{9}$; to this add $\frac{1}{9}$ and we obtain $6\frac{1}{9}=\frac{55}{9}$.

In a similar manner any mixed number can be expressed as an improper fraction.

EXERCISES. VI.

1. Arrange in order of magnitude the fractions $\frac{3}{5}$, $\frac{3}{10}$, $\frac{7}{15}$, and $\frac{9}{19}$, and find the difference between the greatest and the least.

2. Arrange in order of magnitude the fractions $\frac{5}{8}$, $\frac{11}{16}$, $\frac{4}{7}$, and $\frac{7}{10}$, and find the difference between the greatest and the least.

Add together

3. $\frac{1}{2}$, $\frac{1}{8}$, $\frac{1}{7}$, $\frac{1}{14}$, and $\frac{9}{56}$.

4. $\frac{4}{13}$, $\frac{7}{15}$, $\frac{7}{39}$, and $\frac{3}{65}$.

5. $7\frac{11}{18}$, $\frac{17}{24}$, $5\frac{4}{9}$, and $11\frac{11}{36}$.

6. What is meant by (*a*) the numerator; (*b*) the denominator of a fraction? Prove that $\frac{3}{4}=\frac{6}{8}$, and add together $\frac{3}{10}+\frac{7}{12}+\frac{6}{24}$.

Add together

7. $3\frac{7}{15}$, $\frac{13}{16}$, $1\frac{11}{24}$, and $\frac{8}{45}$.

8. $2\frac{5}{18}$, $\frac{16}{21}$, $4\frac{15}{28}$, and $\frac{19}{63}$.

9. $3\frac{11}{12}$, $1\frac{5}{8}$, $4\frac{11}{16}$, and $2\frac{1}{3}$.

10. What number added to $\frac{27}{32}+\frac{15}{56}$ will make $3\frac{9}{224}$?

11. A man leaves £32,818 to be divided among his four sons in the proportions of the fractions $\frac{2}{3}$, $\frac{3}{4}$, $\frac{4}{5}$, and $\frac{5}{6}$. Find the share of each.

12. A certain fraction added to the sum of $\frac{7}{33}+\frac{5}{12}+\frac{2}{11}+\frac{13}{28}$ yielded the total $1\frac{1003}{1848}$. What was this number?

13. Add together $\frac{3}{4}$, $\frac{4}{5}$, $\frac{5}{6}$, and $\frac{7}{8}$.

14. Arrange in order of magnitude $\frac{1}{9}$, $\frac{80}{81}$, $\frac{4}{27}$, $\frac{11}{12}$.

15. A cistern can be filled by one pipe in 20 minutes and by another in 30 minutes; in what time would both together fill it?

16. If I give away $\frac{2}{3}$ of my money, then $\frac{3}{4}$ of what remains, then $\frac{4}{5}$ of the remainder, what fraction of the whole have I left?

17. Three pipes empty a vessel in 6, 4, and 2 hours respectively; how long will they take, all together?

18. Add together $3\frac{2}{7}$, $4\frac{1}{3}$, $\frac{1}{6}$, and $\frac{16}{21}$.

19. A man owns $\frac{1}{5}$ of a ship worth £3484, which is insured for $91\frac{2}{3}$ per cent. of its real value; what would he lose in case of the ship being lost?

Multiplication.—*To multiply a fraction by any whole number, either multiply its numerator by the number or divide its denominator by the number:* thus, if the fraction be $\frac{3}{8}$ and we require to multiply this by 4. As already explained $\frac{3}{8}$ means that the unit is divided into 8 equal parts and 3 of these equal parts are taken. Also 3 such parts multiplied by 4 will give 12 parts, that is, 12 eighths of the unit,

$$\therefore\ \tfrac{3}{8}\times 4=\tfrac{12}{8}.$$

It is easy to prove the above by reference to the foot-rule (Fig. 32), also to show that the result would be the same if, instead of multiplying the numerator by 4, the denominator were divided by that number.

Division.—*To divide a fraction by a whole number either divide the numerator or multiply the denominator by the number.*

Ex. 1. To divide the fraction $\frac{3}{8}$ by 4.

As shown on p. 45 $\frac{3}{8}=\frac{12}{32}$, the latter fraction meaning that the unit is divided into 32 equal parts and 12 of these parts are taken; dividing these 12 parts by 4 we obtain $\frac{3}{32}$.

$$\therefore\ \tfrac{3}{8}\div 4=\tfrac{3}{32};$$

This again can and should be verified by means of the foot-rule.

To multiply a fraction by a fraction.—The rule is: *Multiply the numerators together for a new numerator and the denominators to obtain a new denominator.* This can be shown to be true in any particular case, as in the following:

Ex. 1. $\frac{3}{8}\times\frac{1}{2}$, or as it is sometimes written $\frac{3}{8}$ of $\frac{1}{2}$. Again using the foot-rule (Fig. 32), as the distance from a to b is the unit, the distance ac will be represented by $\frac{1}{2}$. And we are directed to take $\frac{3}{8}$ of this distance, but $\frac{3}{8}$ of this half unit is $\frac{3}{16}$ of the whole distance; hence $\frac{3}{8}$ of $\frac{1}{2}=\frac{3}{16}$.

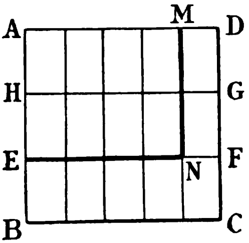


FIG. 33.—To show that $\frac{4}{5}$ of $\frac{2}{3}=\frac{8}{15}$.

Ex. 2. To show that $\frac{4}{5}$ of $\frac{2}{3}=\frac{8}{15}$.

Construct a square *ABCD* of 5 units side (Fig. 33). Divide the base *BC* into 5 equal parts and through the points of division draw lines at right angles to *BC* as shown. Divide *AB* into 3 equal parts. Then *AMNE* is $\frac{4}{5}$ of *ADFE*. But *ADFE* is $\frac{2}{3}$ of the square *ADCB*. Hence *AMNE* is $\frac{4}{5}$ of $\frac{2}{3}$ of the square. But from the figure, since the square is divided into 15 equal parts, and *AMNE* contains 8 of these parts, hence $AMNE=\frac{8}{15}\,ADCB$,

$$\therefore\ \tfrac{4}{5}\text{ of }\tfrac{2}{3}=\tfrac{8}{15}.$$

In a similar manner when three or more fractions have to be multiplied together the product of all the numerators and of all the denominators is obtained, these are used as the numerator and denominator respectively of the required fraction.

Ex. 3. $\frac{24}{\cancel{17}} \times \frac{\cancel{17}}{\cancel{7}} \times \frac{\cancel{7}}{4} = \frac{24}{4} = 6.$

Cancelling.—In the above example by *cancelling* common factors the work is performed in a better and simpler manner than by attempting to multiply all the numerators together to form a new numerator, and the denominators together to form a new denominator.

Division of a fraction by a fraction.—*To divide one fraction by another invert the divisor and then multiply.* As division is the opposite of multiplication, so $\frac{3}{4}$ divided by $\frac{2}{8}$, written $\frac{3}{4} \div \frac{2}{8}$, means such a fraction, which, multiplied by $\frac{2}{8}$, will give as a result $\frac{3}{4}$, and the number is $\frac{3 \times 8}{4 \times 2} = \frac{3}{1}$, or 3.

Simplification of fractional expressions.—The simplification of fractions, although in some cases rather tedious, is of great importance. The following examples will serve to illustrate some of the methods adopted.

Ex. 4. $\frac{1}{2} + \frac{1}{3} + \frac{1}{4}.$
Here the equivalent fractions become $\frac{6}{12} + \frac{4}{12} + \frac{3}{12} = \frac{13}{12}$, and the sum is obtained by adding the numerators.

Ex. 5. Which of the two fractions $\frac{5}{7}$ and $\frac{3}{5}$ is the greater?
The first is equal to $\frac{25}{35}$, the second to $\frac{21}{35}$.
Hence the first is the greater.

Ex. 6. Simplify

$$\frac{\frac{1}{5} - \frac{1}{6} + \frac{1}{9} - \frac{1}{18}}{\frac{1}{4} - \frac{1}{5} + \frac{1}{8} - \frac{1}{9} + \frac{1}{16}} = \frac{\frac{1}{30} + \frac{1}{18}}{\frac{1}{20} + \frac{1}{72} + \frac{1}{16}} = \frac{\frac{3+5}{90}}{\frac{36+10+45}{720}}$$

$$= \frac{720 \times 8}{90 \times 91} = \frac{64}{91}.$$

Ex. 7. Simplify

$$\frac{\frac{9}{11} \text{ of } \left(\frac{41}{162}-\frac{9}{49}-\frac{3}{54}\right)}{\frac{4}{9}+\frac{1}{2}-\frac{13}{14}} \text{ of } \left(4\tfrac{2}{5}-2\tfrac{1}{2}+\frac{3}{10}\right)$$

$$\frac{\frac{9}{11}\times\left(\frac{41}{162}-\frac{9}{162}-\frac{9}{49}\right)}{\frac{17}{18}-\frac{13}{14}}\times\left(2\tfrac{7}{10}-\frac{1}{2}\right)$$

$$=\frac{\frac{9}{11}\times\left(\frac{16}{81}-\frac{9}{49}\right)}{\frac{119-117}{7\times 18}}\times\frac{11}{5}$$

$$=\frac{\frac{9}{5}\times\left(\frac{16}{81}-\frac{9}{49}\right)\times 7\times 18}{2}=1\tfrac{4}{7}.$$

EXERCISES. VII.

1. Multiply $1-\dfrac{\frac{3}{10}}{\left(\frac{1}{2}+\frac{1}{5}\right)}$ by $1-\dfrac{\frac{1}{10}}{\frac{1}{4}-\frac{1}{10}+\frac{1}{25}}$

2. Divide $4\tfrac{9}{10}$ by $12\tfrac{5}{6}$.

Reduce to their simplest forms :

3. $\dfrac{6\frac{1}{4}-3\frac{1}{16}}{3\frac{2}{5} \text{ of } 1\frac{1}{4}}\div\dfrac{12\frac{4}{15}-11\frac{2}{3}}{3\frac{1}{3} \text{ of } \dfrac{1}{4\frac{1}{6}}}$.

4. $\dfrac{1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}}{1+\frac{1}{2}-\frac{1}{3}+\frac{1}{4}}\div\frac{2}{17}$.

5. $\frac{1}{39}$ of $4\frac{1}{3}$ of $\dfrac{\frac{1}{2}+\frac{1}{3}+\frac{1}{4}}{\frac{1}{4}+\frac{1}{5}+\frac{1}{6}}$.

6. $\dfrac{\frac{1}{9} \text{ of } 3\frac{6}{13}+\dfrac{2}{3\frac{1}{4}}-\dfrac{1}{6\frac{1}{2}}}{5+\dfrac{1}{2\frac{1}{6}}+\frac{3}{13} \text{ of } 2\frac{1}{3}}$.

7. $\dfrac{3\frac{1}{4} \text{ of } 4\frac{1}{3}}{\left(2\frac{1}{2}-\frac{1}{3}\right) \text{ of } \left(3\frac{1}{2}-\frac{1}{4}\right)}$.

8. Simplify $\dfrac{\frac{3}{4}+\frac{2}{5}}{\frac{3}{4}-\frac{2}{5}}\times\dfrac{2\frac{2}{3}}{\frac{2}{3}+\frac{3}{7}}$.

9. Simplify $\dfrac{\frac{1}{5}-\frac{1}{6}+\frac{1}{9}-\frac{1}{18}}{\frac{1}{4}-\frac{1}{5}+\frac{1}{8}-\frac{1}{9}+\frac{1}{16}}\times\frac{91}{64}$.

10. The area of a country is 32,300,000 acres. It consists of three kinds of land, viz., first, arable and garden; secondly, meadow, pasture, and marsh; thirdly, waste: the area of these kinds are respectively in proportion to the numbers 2, 3, and $\frac{2}{3}$. How many acres are there of each kind?

11. Find out which is the greater of $\frac{56}{143}$ and $\frac{72}{187}$, and divide $\frac{16}{11}$ by their difference.

12. A sovereign consists of 22 parts by weight of pure gold to 2 parts of alloy, and it weighs 123·274 grains troy. Neglecting the value of the alloy, what is the value of pure gold per ounce troy to the nearest penny?

13. A man pays away half his money to *A*, a third of what he has left to *B*, and a fifth part of what he still has left to *C*; if after these payments he has 12s. 8d. left, how much had he at first?

14. Divide £4. 2s. 6d. amongst three persons, *A*, *B*, and *C*, so that *A*'s share may be $\frac{5}{8}$ths of *B*'s and *B*'s share $1\frac{1}{2}$ of *C*'s.

15. A man owned $\frac{13}{17}$ of a ship; he sold $\frac{5}{8}$ of his share, and then $\frac{3}{4}$ of the remainder. The value of the remainder was £130. Find the value of the whole ship.

16. A man owned $\frac{1}{3}$ of a ship, and after buying additional shares equal to $\frac{3}{5}$ of the ship, his total interest in her was worth £28,700; what was the value of the ship?

Decimal Fractions.—On examining the number 5555 (five thousand five hundred and fifty-five) it is seen that the same figure is used throughout, but the value of different fives is very unequal. For every place removed to the left the value is increased ten times. Or, in reading a set of figures from left to right, each figure has one-tenth the value it would have if it were moved one place to the left. Decimal fractions are obtained by using the same notation to indicate numbers less than unity, each digit having one-tenth of the value that it would have if it stood one place farther to the left.

Thus 5·555 means $5+\frac{5}{10}+\frac{5}{100}+\frac{5}{1000}$; the point used in this notation is called the decimal point, or simply the point, and the above number would be read as "five, point, five, five, five." In a similar manner, 8·073 would be read as *eight*, *point*, *nought*, *seven*, *three*.

Addition and Subtraction of Decimals.—When decimal fractions are used, the simple rules of Arithmetic can be applied to any given example readily and easily. Addition and subtraction are performed exactly as in the case of whole numbers, the only precaution necessary to prevent mistakes is to keep the decimal points under each other.

Multiplication of Decimals.—The process of multiplication is carried out exactly as in the case of whole numbers; to obtain the decimal place in the product, commencing from the right, as many digits are pointed off as there are digits following the decimal point in the multiplier and the multiplicand added together.

Ex. 1. 36·42 × 4·7.

Multiplying 3642 by 47, we obtain the product 171174. As there are two digits following the decimal point in the multiplicand, and one digit following the decimal point in the multiplier, we point off three digits, giving as a result 171·174.

Ex. 2. ·000025 × ·005.

Here 25 × 5 = 125.

In the multiplicand there are six digits following the decimal point, and in the multiplier three. Hence the product is ·000000125. The six digits and the three digits referred to are often referred to as "decimal places"; thus ·000025 would be said to consist of six decimal places.

EXERCISES. VIII.

Multiply:

1. 168·05 by 1·0069.
2. 67·891 by ·04508.
3. 4·86923 by ·007655.
4. 8·0198 by ·001077.
5. 500·173 by ·00376.
6. 709·285 by 1·0069.
7. 321·4379 by 3·063.
8. 709·285 by 4·0507.
9. 67·087 by 3·6037.
10. 28·3075 by ·00894.
11. 24·9735 by 30·307.
12. 473·21 by 11·00735.
13. 13·0765 by 6·80704.
14. 25·6098 by ·03572.
15. 11·82075 by ·003961.
16. 39·0437 by 5·40305.

Division of Decimals.—The division of one quantity by another when decimals enter into the operation, is performed exactly as in the case of whole numbers. The process can be best explained by an example as follows:

Ex. 1. Divide ·7 by ·176.

This may be described as finding a number, such, that when multiplied by ·176, the product is equal to ·7.

Decimals may be divided as in the case of whole numbers, but care is necessary in marking off the decimal point. In the present and in all simple cases, the position of the decimal point is evident on inspection. Practically, it is often convenient to multiply both terms by 10, or some multiple of 10—100, etc.—and so obtain at once without error the position of the unit's figure, and hence of the decimal point.

Thus, in the above example, multiplying both terms by 10, we have to divide 7 by 1·76, and it is at once seen that the number required lies between 3 and 4; this determines the position of the unit's figure; as 7·0 is unaltered by adding any number of ciphers to the right, we add two for the purpose of the division, and multiplying 1·76 by 3 we obtain 5·28, which, subtracted from 7·00, gives a remainder 1·72; to this we affix a cipher and carry on the division as far as necessary; when this is done, we find ·7÷·176 =3·9772727.

$$\begin{array}{r} 1{\cdot}76\,)\,7{\cdot}00\,(\,3{\cdot}97 \\ \underline{5\;28}\phantom{\,(\,3{\cdot}97} \\ 1\;720\phantom{(\,3{\cdot}97} \\ \underline{1\;584}\phantom{(\,3{\cdot}97} \\ 1360\phantom{\,3{\cdot}97} \\ \underline{1232}\phantom{\,3{\cdot}97} \\ 1280\phantom{3{\cdot}97} \end{array}$$

It will be seen that the ordinary method of performing division necessarily requires considerable space, especially when there are several terms in the quotient. Another method in which only the results of the several subtractions are written down is often used, the method of procedure is as follows:—Note, as before, that 1·76 will divide into 7; then since 3×6=18, the 8 is not written down but instead it is mentally subtracted from 10 giving 2. Next 3×7=21 and 1 to carry makes 22, the 2 is again not written down, but, instead after the addition of unity from last process we say 3 from 10=7. In a similar manner the remaining figure is obtained; the next row of figures is arrived at in like manner and so on. Comparing the two examples it will be seen that at each step of the work one line of figures is dispensed with.

$$\begin{array}{r} 1{\cdot}76\,)\,\underline{7{\cdot}00}\,(\,3{\cdot}97 \\ \underline{1\;720}\phantom{(\,3{\cdot}97} \\ \underline{1360}\phantom{\,3{\cdot}97} \\ 1280\phantom{3{\cdot}97} \end{array}$$

The above example illustrates another important point, viz., that although the quotient may in some cases consist of many digits following the decimal point, it may come to an end eventually; but in other cases, the quotient goes on without end, and is known as a *recurring* decimal. It is not necessary in practical calculations to continue any operation in decimals to more than the third or fourth place, and in the majority of cases, if the quotient is correct to the second decimal place, it is amply sufficient. In the previous example, the quotient correct to the second decimal place would be 3·98. If the next figure to the 7 had been less than 5, it would not have been necessary to add to the preceding figure; but when the figure rejected is 5, or greater than 5, the previous figure is increased by unity as shown.

It is obviously bad in principle to use more figures than are necessary for the work in hand; these are not only unnecessary, but give additional trouble, and also increase the risk of making mistakes. In many cases, students are found to work with ten or more decimal figures, when, owing to errors of observation or to slightly incorrect data, even the first decimal place may not be reliable. It is, of course, inadvisable to add an error of arithmetic to an uncertainty of measurement or data, but even a slight error is preferable to working out ten or fifteen places of decimals to a practical question when one decimal place at most is sufficient.

EXERCISES. IX.

Divide:

1. ·39908027 by 248·03.
2. ·4990107712 by ·00608.
3. 2515·611 by ·0785.
4. 360·221286 by 898·98.
5. 79682·44 by ·0172.
6. 1·17597 by 18·07.
7. 1·047034 by ·0302.
8. 2·5 by 32.
9. ·0325 by 6·5.
10. 155·73 by 3580.
11. 124·5 by ·0083.
12. ·0141009 by 47·003.
13. 9·225 by ·00375.
14. 470·6099645 by ·1235.
15. 4·0975 by ·2980.
16. ·15104932 by 40·13.

Conversion of a vulgar into a decimal fraction.—To convert a vulgar fraction to a decimal fraction, reduce the vulgar fraction to its lowest terms and then divide numerator by denominator.

Ex. 1. $\frac{3}{8}=3\div8=\cdot375$; $\frac{7}{8}=7\div8=\cdot875$.

Ex. 2. $\frac{1}{3}=\cdot333\ldots$.

The result of the division is shown by as many threes as we care to write. The notation $\cdot\dot{3}$ is used to denote this unending row.

Again $\frac{2}{3}=\cdot666\ldots=\cdot\dot{6}$.

In each of these and in similar cases the equivalent vulgar fractions are obtained by writing 9 instead of 10 in the denominator, thus $\cdot\dot{3}=\frac{3}{9}=\frac{1}{3}$, etc.

When necessary to add or subtract recurring decimals, as many of the recurring figures as are necessary for the purpose in hand are written and the addition or subtraction performed in the usual manner.

In a similar manner $\frac{1}{7} = \cdot 142857$, and these figures again recur over and over again as the division proceeds, hence $\frac{1}{7} = \cdot\dot{1}4285\dot{7}$. With a little practice the student soon becomes familiar with the more common recurring decimals, and rules are neither necessary nor desirable in such a book as this.

Decimals of concrete quantities.—It is often necessary to express a given quantity as a fraction of another given quantity of the same kind. Thus, in the case of £1. 15s, it is obvious that 15s. $= \frac{15}{20}$ of 20 shillings, and thus £1. 15s. may be written £$1\frac{3}{4}$, or as $\frac{3}{4} = \cdot 75$ as £1·75.

Ex. 1. To reduce 10d. to the decimal of a pound.

As there are 240 pence in £1,

$$\therefore \text{ required fraction is } \tfrac{10}{240} = \tfrac{1}{24} = £\cdot 0416 \ldots .$$

Ex. 2. Express 6 days 8 hours as the decimal of a week.

$$6 \text{ days } 8 \text{ hours} = 6\tfrac{8}{24} = 6\tfrac{1}{3} \text{ days},$$

$$\therefore\ 6 \text{ days } 8 \text{ hours} = \frac{6\cdot\dot{3}}{7} = \cdot 90476\dot{1} \text{ week}.$$

Ex. 3. Reduce 5d. to the decimal of 1s.

$$\tfrac{5}{12} = \cdot 41\dot{6}.$$

In many cases it will be found simpler and easier to reduce a fraction to its equivalent decimal if the numerator and denominator are multiplied by some suitable number.

Ex. 4. Reduce $\frac{7}{250}$ to a decimal.

Multiplying by 4 we get $\frac{28}{1000} = \cdot 028$.

In a similar manner $\frac{1}{25} = \frac{4}{100} = \cdot 04$.

Other examples can be worked in like manner.

Given a decimal fraction of a concrete quantity, its value can be obtained by the converse operation to that described.

Ex. 5. Find the value of ·329 of £1.

The process is as follows: First multiplying by 20 we obtain the product 6580, and marking off three decimals we get the value 6·580 shillings. In a similar manner multiplying by 12 and 4 as shown, we obtain the value of ·329 of £1, which is read as 6 shillings 6 pence 3 farthings and ·84 of a farthing.

$$\begin{array}{r} \cdot 329 \\ 20 \\ \hline 6\cdot 580 \\ 12 \\ \hline 6\cdot 960 \\ 4 \\ \hline 3\cdot 840 \end{array}$$

The result could be obtained also by multiplying ·329 by 240, the number of pence in £1, giving 78·96d. and afterwards reducing to shillings, etc.

EXERCISES. X.

Reduce

1. ·00233 of a ton to lbs. and the decimal of a lb.

2. ·0925 of £1. 4s. to pence and the decimal of a penny.

3. ·03042 of 3 tons 15 cwt. 75 lbs. to ounces and the decimal of an ounce.

4. What decimal of a sovereign is ·215 of £4. 16s.?

5. Reduce 1·095 of 3 days 13 hours 43 minutes to minutes and the decimal of a minute.

6. Reduce 5·067 of 1 week 3 days 13 hours to minutes and the decimal of a minute.

7. Add together ·0185 of a week and ·634 of an hour.

8. Reduce 2·086 of 5 days 19 hours 37 mins. to seconds and the decimal of a second.

9. Subtract ·01547 of $1\frac{6}{7}$ weeks from 12·65083 hours, and give the answer in minutes and the decimal of a minute.

10. Add together ·006 of a ton, ·03 of a cwt., and ·09 of 3 quarters, and express the result in lbs. and decimals of a lb.

11. Express £3. 6s. 6d. as a decimal fraction of 7 half-crowns.

12. Add together ·032 of a pound, 2·35 of a shilling, and 5·42 of a penny, and subtract ·35 of half a crown from their sum. Express the answer in pence and decimals of a penny.

MISCELLANEOUS EXERCISES. XI.

1. State the rule for the division of one decimal fraction by another, and find the first five decimal places of the quotient of 0·1373 divided by 0·021.

Perform the operations indicated in the following:

2. $(21{\cdot}7 \times 0{\cdot}087 \div 2{\cdot}03) + (102{\cdot}01 \times 0{\cdot}319 \div 2{\cdot}639)$.

3. $(7{\cdot}35 \times 0{\cdot}0143 \div 15{\cdot}015) - (0{\cdot}152 \times 0{\cdot}033 \div 2{\cdot}09)$.

4. $(18{\cdot}7 \times 0{\cdot}0039 \div 2{\cdot}21) - (0{\cdot}441 \times 0{\cdot}0091 \div 1{\cdot}911)$.

5. $(1{\cdot}61 \times 0{\cdot}0209 \div 0{\cdot}00253) - (2{\cdot}03 \times 0{\cdot}336 \div 32{\cdot}48)$.

6. $(0{\cdot}0057 \times 2{\cdot}09 \div 0{\cdot}361) - (0{\cdot}00165 \times 0{\cdot}077 \div 0{\cdot}0105)$.

7. $(0{\cdot}0011 \times 0{\cdot}091 \div 0{\cdot}0035) + (0{\cdot}015 \times 0{\cdot}507 \div 0{\cdot}39)$.

8. $(0{\cdot}1 \times 0{\cdot}01 \div 0{\cdot}0002) - 0{\cdot}6375 \times \frac{1}{1{\cdot}7} \div 0{\cdot}125)$.

9. Reduce the following fraction to the form of a decimal true to three places:

$$\frac{\frac{3}{11} \times 25{\cdot}15}{4\frac{6}{11} \times 0{\cdot}4}.$$

10. Divide $0{\cdot}2 \times 0{\cdot}3 \div 0{\cdot}00012$ by $\left(5{\cdot}719 \times \frac{1}{1{\cdot}9}\right) + (0{\cdot}3 \times 1{\cdot}2 \times 9)$.

11. Express the length of a seconds pendulum (39·1393 in.) as the decimal of a metre, correct to 4 places. (1 metre = 39·371 inches.)

12. Reduce to the simplest form:

$$\text{(i)}\quad \left(\tfrac{3}{4} \text{ of } \tfrac{12}{11} - \tfrac{1}{2}\right) \times \frac{1 + \left(\frac{2}{3} \text{ of } \frac{39}{49}\right) \div \frac{78}{84}}{2 + \frac{1}{4} + \frac{1}{6} - \frac{13}{9} \div \frac{11}{18}}.$$

$$\text{(ii)}\quad \frac{1{\cdot}1931 \div 5{\cdot}82}{0{\cdot}225 + 0{\cdot}2666\ldots}.$$

13. (i) Express 6 cwts. 3 qrs. $3\frac{1}{2}$ lbs. as a decimal of a ton.
(ii) Find what decimal part of £78. 8s. 8d. is equal to £63. 14s. $6\frac{1}{2}$d.

Averages.—What is called the average of a series of quantities is often wanted, and the student may require to use the term average when applied to various values. Thus, for example, if a railway train is found to complete a journey of 60 miles in $1\frac{1}{2}$ hours, we say that the average speed is 40 miles per hour; meaning that a train moving uniformly at such a speed would move over 60 miles in the given time.

Again, if in a boat's crew of 8 men the weights of the men are respectively 12, 13, 10, 8, 9, 14, 15, and 11 stones. Then the total weight

$$= 12 + 13 + 10 + 8 + 9 + 14 + 15 + 11 = 92,$$

and average $= \frac{92}{8} = 11\frac{1}{2}$ stones.

Other simple instances could be quoted, the method in each and all cases being the same. Namely, to obtain the average, first find the total of the given similar quantities and by dividing the result by the number of quantities the average is obtained.

EXERCISES. XII.

1. The weights of a boat's crew are 10 st. 9 lbs., 11 st. $9\frac{3}{4}$ lbs., 12 st. 6 lbs., 11 st. $13\frac{1}{2}$ lbs., 12 st. $8\frac{1}{2}$ lbs., 12 st. $10\frac{1}{2}$ lbs., 12 st. $4\frac{1}{2}$ lbs., 10 st. $12\frac{3}{4}$ lbs. Find the average weight.

2. In testing a machine the following values of the efficiency were found; find the average efficiency: 93·67, 62·27, 77·84, 71·82, 68·09, 65·5, 63·83.

3. What is the average of the following: ·2207, ·2602, ·291, ·311, ·330, ·345?

4. The population of two counties are 2,920,485 and 176,323, the corresponding poor rates are £2,305,708 and £77,425; find the average poor rate per head in each county.

5. In a certain street there are 245 houses; of these 47 have each three families, 54 have 4, and the remainder 2, residing in each; supposing the families to average 4 persons, what is the population of the street?

Summary.

Prime numbers are those numbers which are divisible by no other number except unity.

Measure or Factor.—A number contained an exact number of times in one or more numbers is said to be a measure or factor of those numbers.

Greatest Common Measure.—The greatest number which is contained an exact number of times in each of two or more given numbers is called their G.C.M. It is often also called the Highest Common Factor, or, H.C.F.

Least Common Multiple.—The smallest number which contains each of two or more numbers is called the Least Common Multiple of the numbers.

A **Fraction** is a part or parts of a whole.

To add or subtract Fractions.—Reduce, if necessary, to a common denominator, then add or subtract as required and place the sum or difference over a common denominator.

To multiply fractions.—Multiply the numerators together for a new numerator and the denominators for a new denominator.

To divide fractions.—Reduce, if necessary, both fractions to a simple form, invert the divisor, and proceed as in multiplication.

To reduce a vulgar fraction to its lowest terms.—Divide both numerator and denominator by any factors common to both, or find the G.C.M. of numerator and denominator, and divide both by it.

Decimals.—Addition and subtraction are performed as in simple addition and subtraction, the figures are set down so that the decimal points are one above the other.

Multiplication.—Proceed as in the case of whole numbers, and point off as many decimal places as there are in multiplier and multiplicand together.

Division.—Proceed as in whole number, and point off as many decimal places as those in the dividend exceed those in the divisor.

To convert a vulgar to a decimal fraction.—Divide the numerator by the denominator; add decimal point as required.

CHAPTER V.

INVOLUTION AND EVOLUTION.

SIMPLE APPLICATIONS OF ARITHMETICAL RULES.

Contracted Multiplication and Division.—The results of all experiments show at best only an approximation to a true result. The accuracy of an expression is, it is true, increased by extending the number of decimal figures in the result, but it should be carefully noted that the accuracy of any result does not depend on the number of decimal places to which the result is calculated, but on the accuracy with which the observations were made.

In any result obtained the last decimal place may not be accurate, but the figure preceding should be as accurate as possible. It is advisable for the sake of accuracy to carry the result to one place more than is required in the result.

It will at once be evident that to multiply together two numbers (in each of which several decimal figures occur), and afterwards to reject several decimals from the product, loss of time will be experienced. Especially is this the case in practical questions in which the result is only required to be correct to the second or third place of decimals; in such cases what is known as Contracted Multiplication is useful.

Contracted Multiplication.—In this method the multiplication by the *highest* figure of the multiplier is first performed. By this means the first partial product obtained is the most important one.

The method can be shown and best understood by an example.

Ex. 1. Multiply ·006914 by 8·652.

The product of the two numbers can of course be obtained by the ordinary methods; to compare the two methods, "ordinary" and "contracted," the product is obtained as shown:

Ordinary Method.	*Contracted Method.*
6914	6914
8652	2568
13828	55312
34570	$4148\cancel{4}$
41484	$346\cancel{7}\cancel{0}$
55312	$14\cancel{8}\cancel{2}\cancel{8}$
·059819928	·059820

The ordinary method will be easily made out from the working shown; in the contracted method it will be seen that **the figures in the multiplier are reversed**, and the process continued as follows: Multiply first by 8, so obtaining 55312; next by 6—this step we will follow in detail—6 × 4 = 24, the 4 is not written down (but if written down it is cancelled as shown), and the 2 is carried on. Continuing, 6 × 1 = 6, and adding on 2 gives 8. Next, 6 × 9 = 54, the 4 is entered; and 6 × 6 is 36, this with the 5 from the preceding figure gives 41, hence the four figures are 8, 4, 1, 4 as shown.

In the next line, multiplying by 5, we can obtain the two figures 0 and 7 as shown, but as these are not required unless there is some number to be carried, it is only necessary to obtain 69 × 5, and write down the product 345, add 1 for the figure rejected making 346 as shown; finally, as 2 × 9 will give 18, and therefore we have to carry 1, we obtain 2 × 6 = 12, together with the 1 carried from the preceding figure gives 13, add 1 gives 14. Adding all these partial products together we obtain the product required.

Thus in the second row one figure is rejected, in the next row two figures, and in the last row three figures are left out.

It must be noticed that when the rejected figure is 5 or greater, the preceding figure is increased by 1, also that the last figure of the product is not reliable. Having noted (or cancelled) the rejected figures, as will be seen from the example, the decimal point is inserted as in the ordinary method. The position of the decimal point can, in the majority of cases, be effected by mere inspection, and the ability to ascertain without error the position of the decimal point is necessary when the slide-rule is used. The student is advised to get the practice necessary to do this without effort.

Contracted Division.—It is assumed that the student is familiar with the ordinary method of obtaining the quotient in the case of division, but this long process of division can also be advantageously contracted. The method of doing this will be clear from the following worked example.

Ex. 1. Divide ·03168 by 4·208.

We shall work this example by the contracted method alone.

$$\begin{array}{r@{\,}r@{\,}l}
4208\,) & 31680 & (\,7529 \\
 & \underline{29456} & \\
 & 2224 & \\
 & \underline{2104} & \\
 & 120 & \\
 & \underline{84} & \\
 & 36 & \\
 & \underline{\underline{36}} &
\end{array}$$

In the above example the number 7 is obtained by the usual process of division. By multiplying the divisor by 7, the product 29456 is arrived at. When this is subtracted from 31680, the remainder 2224 is left. It is seen that if we drop or cancel the 8 from the divisor 4208, thus obtaining 420, it can be divided into the remainder 2224 five times. In multiplying by 5 we take account of the 8 thus, as 5×8 is 40, we do not enter the 0 but carry on the 4; but $0 \times 5 = 0$, and adding 4, we see this is the figure to be entered. Now proceed to the next and the following figures, obtaining in the usual way 2104; subtract this from 2224, and the remainder 120 is obtained. Proceeding in like manner with the multiplier 2, we obtain 84, which, subtracted from 120, leaves 36, and our last figure in the quotient is 9. By the method described on p. 52, the answer is ·007529.

The above example shows that the method of contracted division consists in leaving out or, as it is called, rejecting a figure at each operation. Any number which would be added on to the next figure by the multiplication of the rejected figure is carried forward in the usual way. To avoid mistakes it may be convenient to either draw a line through each rejected figure of the divisor, or to place a dot under it.

EXERCISES. XIII.

1. Without introducing into the work any unnecessary figures, find, to four significant figures, the values of ·01785 × 87·29 and 1·046 ÷ 25·87.

2. ·0008679 × 496·038.

3. 12·607 × 3·08 × 1·0006.

4. 3·0964515 ÷ ·0645.

5. 589·0067 × 3·1008.

6. 10·834394 ÷ ·05309.

7. 18596·508 ÷ 98760.

8. 8561·02 × 5·6039.

9. 4·37642 × ·00172.

10. 1·6308 ÷ 362·4 × 1·942.

11. 7·0342 × ·003206.

12. 5·61023 × ·597001 ÷ ·001.

Involution.—When a number is multiplied by itself once, twice, or more times, the process is called *Involution.* The number thus multiplied is called the **root,** and the products are called the **powers** of the number. Any number multiplied by itself once is said to be **squared,** and the product so obtained is called the **square** or the **second power** of the original number. The number itself is called the **square root** of the product.

Thus, $3 \times 3 = 9$. Here the product 9 is the square or second power of 3, and 3 is the square root of 9.

Instead of writing the expression 3×3 a small figure is placed near the top of the number or quantity, and on the right-hand side of it, thus: 3^2. This indicates how many times the number appears in the product. Thus, we write 3×3 as 3^2, $3 \times 3 \times 3$ as 3^3, etc. The smaller figure written near the top of a number in the manner described is called the **index** or *exponent* of the number.

Adopting this notation, 3^1 would be called the first power of 3, 3^2 the second power, 3^3 the third power, etc.

The squares, cubes, or even higher powers can be easily obtained if the number is not greater than 10 (higher powers are best obtained by using logarithms). Thus $2^2=4$, $2^3=8$, $2^4=16$, etc.

The powers of 10 itself are easily remembered, and are as follows: $10^2=100$, $10^3=1000$, $10^6=1{,}000{,}000$, etc.

$$10^{-1}=\tfrac{1}{10},\ 10^{-2}=\tfrac{1}{100},\ 10^{-3}=\tfrac{1}{1000},\ \text{etc.}$$

This method of indicating large numbers is very convenient in Physical Science, in which such numbers as 2, 5, or 10 millions, etc., are of frequent occurrence; for in place of writing

5,000,000 for instance, we may write it more shortly as 5×10^6; and, as shown in the chapter on logarithms (p. 148), we can more easily write down the characteristic of the logarithm of the number.

In general terms it may be said that the powers of any quantity a would be written as a^2, a^3, a^4, etc.

The squares of all the numbers from 1 to 10 should be remembered; they are as follows:

$$1^2=1, \quad 4^2=16, \quad 7^2=49,$$
$$2^2=4, \quad 5^2=25, \quad 8^2=64,$$
$$3^2=9, \quad 6^2=36, \quad 9^2=81.$$

In a similar manner the squares of all numbers from 10 to 20 should be written down as well as the cubes of all the numbers from 1 to 10 inclusive.

EXERCISES. XIV.

1. Find the square of 25.
2. Find the square of 233.
3. ,, cube of 3·17.
4. ,, cube of 4·95.
5. The square and cube of 37.
6. The square and cube of 9·7.

Evolution.—The reverse of Involution is to extract, or, find the roots of any given powers.

The root of a power or number is such a number, that, multiplied by itself a certain number of times, it will produce that power. Thus the square root of a given number is that number, which, when multiplied by itself, is equal to the given number.

The root of a given number may be denoted by the symbol $\sqrt{}$ placed before it, with a small figure indicating the nature of the root placed in the angle; thus, the cube root of 27 is denoted by $\sqrt[3]{27}$, the fourth root of 64 by $\sqrt[4]{64}$, and so on. The square root in this manner would be denoted by $\sqrt[2]{9}$, but the 2 is usually omitted, and it is written as $\sqrt{9}$.

Another, and for many purposes a better method, is to indicate the root by a fraction placed as an index, and referred to as a **fractional index**. Thus, for example, the square root of 9 is written $9^{\frac{1}{2}}$, and is read as nine to the power one-half. Similarly, the cube root of 27 is written as $27^{\frac{1}{3}}$, meaning 27 to the power one-third.

To obtain the square root of any quantity in cases where it is not possible to ascertain the root by inspection, we have (to avoid unnecessary repetition) to adopt a rule. The following example will illustrate the method of extracting a square root.

Ex. 1. Find the square root of 155236.

$$\begin{array}{rl}
 & 1\dot{5}5\dot{2}3\dot{6}\,(\,300+90+4 \\
 & \underline{90000} \\
2\times300+90=690\,) & 65236 \\
 & \underline{62100} \\
2\times390+4=784\,) & 3136 \\
 & \underline{3136}
\end{array}$$

The process is as follows :

Divide the given number into periods of two figures each, by putting a point over the *unit's figure*; on the figure 2, which is in the second place to the left of the 6 ; and also on the 5, as shown. As the given number consists of six figures, the required square root contains three. As $300^2=90{,}000$ and $400^2=160{,}000$, the required square root lies between 300 and 400 ; hence we put 300 to the right of the given number, and subtract its square 90,000 ; this gives a remainder of 65236.

Put twice 300 to the left of 65236, this divides into 65236 a little over 90 times; add 90 to 2×300, giving 690; this multiplied by 90 gives a product of 62100; subtract this from 65236, and the remainder 3136 is obtained.

Next set down to the left $2\times390=780$; this will divide 4 times into 3136. Add on 4 to 780, giving 784, and multiply by 4 obtaining 3136; this subtracted leaves no remainder; thus 394 is the square root required.

The ordinary practical method is as follows :

$$\begin{array}{rl}
 & 1\dot{5}5\dot{2}3\dot{6}\,(\,394 \\
 & \underline{9} \\
69\,) & 652 \\
 & \underline{621} \\
784\,) & 3136 \\
 & \underline{3136}
\end{array}$$

Point as before, and find the largest number the square of which is less than 15; this is 3. Set the figure to the right of the given number and its square under the first pair of figures, subtract 9 from 15, giving a remainder 6.

Bring down the next two figures 52, giving the number 652.

Next put the double of 3, that is, 6, on the left, and we find on trial that 6 will divide into 65 nine times. Put the 9 on the right and also with the 6, and multiply 69 by 9, giving 621 ; this subtracted from 652 gives a remainder of 31.

Bring down the next two figures, thus obtaining 3136. Double the number 39, the part of the root already found, and put it on the left, as shown.

By trial, we find that 78 will divide into 313 four times ; put the four on the right and also with the 78, making the number 784 ; this, multiplied by 4, the last figure added, gives 3136, which, subtracted, leaves no remainder. Hence 394 is the square root required.

The student should always begin to point at the unit's place, whether the given number consists of integers, or decimals, or both.

Ex. 2. Find the square root of 1481·4801.

$$\begin{array}{rl}
 & 1\dot{4}8\dot{1}\cdot4\dot{8}0\dot{1}\ (\ 38\cdot49 \\
 & \underline{9} \\
68\,) & 581 \\
 & \underline{544} \\
764\,) & 3748 \\
 & \underline{3056} \\
7689\,) & 69201 \\
 & \underline{\underline{69201}}
\end{array}$$

The pointing begins at the unit's place, and every alternate figure to the right and left of the unit's place is marked as shown. As there are two dots to the left of the unit's place, the square root consists of the whole number 38 and the decimal ; the working is exactly the same as in the previous example. It should be carefully noticed that, to obtain the square root of a decimal fraction, the pointing should commence from the second figure of the decimal place.

Ex. 3. Find the square root of $\cdot9\dot{2}16$.

$$\begin{array}{rl}
 & \cdot9\dot{2}1\dot{6}\ (\ \cdot96 \\
 & \underline{81} \\
186\,) & 1116 \\
 & \underline{\underline{1116}}
\end{array}$$

The method adopted will be evident from the working shown.

As examples obtain the square roots of the following frequently occurring numbers, these should be worked out carefully and the results remembered.

$$\sqrt{2} = 1{\cdot}414\ldots,$$
$$\sqrt{3} = 1{\cdot}732\ldots,$$
$$\sqrt{5} = 2{\cdot}236\ldots,$$
$$\sqrt{6} = 2{\cdot}449\ldots.$$

It will be found that the square root of such numbers is an unending decimal. Thus, the square root of 3 can be carried to any number of decimal places, but the operation will not terminate. Such a square root is often called a **surd**, or an **incommensurable** number.

In any practical calculation in which *surds* occur the value is usually not required to more than one or two decimal places.

If a number can be easily separated into factors, the square root can readily be obtained. The method adopted would be to try in succession if the number is divisible by 4, 9, 16, etc.

Ex. 4. To find the square root of 1296.

$$1296 = 4 \times 324 = 4^2 \times 81\,;$$
$$\therefore\ \sqrt{1296} = \sqrt{4^2 \times 81}$$
$$= 4 \times 9 = 36.$$

A similar method may be adopted in the case of numbers, the roots of which cannot be expressed as whole numbers.

Ex. 5.
$$\sqrt{128} = \sqrt{16 \times 4 \times 2}$$
$$= 8\sqrt{2},$$

and remembering that the root of $\sqrt{2}$ is 1·414, approximately, the value $8 \times 1{\cdot}414 = 11{\cdot}312$ can be found.

Ex. 6.
$$\sqrt{243} = \sqrt{81 \times 3}$$
$$= 9\sqrt{3}.$$

EXERCISES. XV.

1. Write down and explain the rule of pointing in the extraction of a square root.

Find the square roots of the following:

2. 197·96492.
3. 63409369.
4. ·02010724.
5. 6·249.
6. 1664232025.
7. 6877219041.
8. 253009.
9. 450643·69.
10. 63409369.
11. 4·38204.
12. $1{\cdot}36\dot{1}$.
13. 50 to four decimal places.
14. $\cdot 0\dot{1}2\dot{6}$ to four decimal places.

Cube Root.—As already explained, the cube of a number is the product obtained by multiplying a number three times by itself. Thus, the cube of 4 is $4 \times 4 \times 4 = 64$, and the cube root of 64 is the number 4.

The cubes of numbers from 1 to 12 should be familiar, and when this is done, the cube root of many numbers can be obtained with little trouble and with sufficient accuracy for all practical purposes.

The arithmetical rule of finding the cube root (of a large number or a number which cannot be obtained by inspection) is a troublesome process, and for all practical purposes is not worth remembering. The cube root is easily obtained by logarithms (p. 151). The cube root is written as $\sqrt[3]{\ }$. Thus, $\sqrt[3]{8} = 2$, $\sqrt[3]{64} = 4$, etc.

A better way to express a cube root is $8^{\frac{1}{3}}$ and $64^{\frac{1}{3}}$.

In a similar manner the third, fourth, or fifth roots of a quantity a would be written as $a^{\frac{1}{3}}$, $a^{\frac{1}{4}}$, $a^{\frac{1}{5}}$ respectively.

When a given number can be separated into factors, the cube root may be found readily.

Ex. 1. Find the cube root of 262144.

We have $262144 = 4096 \times 64 = 16^3 \times 4^3$

$$\therefore \quad \sqrt[3]{262144} = \sqrt{16^3 \times 4^3} = 16 \times 4 = 64.$$

Ex. 2. $\sqrt[3]{1953125} = \sqrt[3]{15625 \times 125} = 25 \times 5 = 125.$

MISCELLANEOUS EXERCISES. XVI.

1. The length of a side of a square is 724 feet; find the length of a diagonal.

2. Find to the nearest integer how many inches there are in the length of one of the sides of a square field, whose area is 2 acres.

3. Extract the square root of 0·029 to five places of decimals.

4. The volume of a cubical block is 2248·091 cubic feet; find the area of one of its faces.

5. Extract the square root of 0·51 and 0·051 to four places of decimals.

6. The sides of a rectangle are 16 feet and 10 feet long respectively; find to four places of decimals the length of a diagonal of a square whose area equals that of the rectangle.

7. Extract the square root of $8\frac{1}{3}$ and 0·144 in both cases to three places of decimals.

Simple Arithmetical Methods.

Many simple labour-saving methods are used by practical men, and are in many cases suggested by the circumstances of the problem. We may illustrate by one or two simple examples.

To multiply by 25. Add 2 cyphers and divide by 4.

Ex. 1. $52420 \times 25 = \frac{5242000}{4} = 1310500.$

To divide by 25. The reason for the rule is evident, since $4 \times 25 = 100$. In a similar manner we proceed to divide by 25 as follows: Multiply the number by 4 and divide by 100.

Ex. 2. $27632 \div 25 = \frac{4 \times 27632}{100} = 1105{\cdot}28.$

To divide by 125. Multiply by 8 and divide by 1000.

Subtraction.—When required to subtract one quantity from another, it is often easier to perform the "complementary" method of addition.

Ex. 3. Subtract 4s. 7½d. from 15s.

In this case we may proceed thus; adding 4½d. to 7½d. we obtain 1s. Mentally this is carried on to the next figure, 4, making it into 5, and to convert 5s. into 15s. we require 10s. Hence, to convert 4s. 7½d. into 15s. we must add 10s. 4½d.; in this manner subtracting 4s. 7½d. from 15s. is converted into the easier mental operation of addition.

The method indicated is very useful when the sum of two or more quantities has to be taken from some known quantity. Thus the three angles of a triangle are known to be equal to 180° (p. 34); it is often necessary to find the third when two are given.

Ex. 4. Two angles of a triangle are 29° 30′ 20″ and 46° 10′ 30″ respectively; find the magnitude of the remaining angle.

By the ordinary method we should first obtain the sum of the two given angles.

$$\text{Thus, } 29^\circ\,30'\,20'' + 46^\circ\,10'\,30'' = 75^\circ\,40'\,50''.$$
$$180^\circ - 75^\circ\,40'\,50'' = 104^\circ\,19'\,10''.$$

Instead of the above method we may proceed as follows:

$$\begin{array}{r} 29^\circ\,30'\,20'' \\ \underline{46^\circ\,10'\,30''} \\ \underline{104^\circ\,19'\,10''} \end{array}$$

Write down the given quantities under each other, as in addition, and proceed to write down such an angle as will make the sum 180°. Thus, adding 30″ and 20″, we require 10″ to make the sum 60″; putting down the 10″, as shown, we carry 1′ to the next, which adds up to 41′; requiring 19′ to bring this to 60′ we write down 19′, and carry one, as before, to the next figure, we find $7+9=16$; requiring 4, carry the 2 we have $6+2=8$. Hence, the result is 104° 19′ 10″, as before. The method indicated will be found very useful and accurate, and much more readily applied than the ordinary method.

Multiplication and Division.—In multiplication and division the contracted methods already referred to will be found very convenient. The work can also be further shortened by using approximate multipliers.

Ex. 5. Find the total width in 20 boards, each 11¾ inches wide.

Here, instead of multiplying $20 \times 11\frac{3}{4}$, we may, with advantage, multiply 20×12, giving 240, and subtract $20 \times \frac{1}{4}$ or 5,

$$\therefore\ 20 \times 11\tfrac{3}{4} = 235.$$

Ex. 6. Find the cost of 30 articles at 1s. 7½d. each.

The cost could be obtained by reducing 1s. 7½d. either to pence or half-pence, multiplying by 30, and afterwards by division, to shillings, etc. But the cost can be obtained mentally, thus:

$$\left.\begin{array}{l} 30 \text{ at } 1\text{s.} = 30\text{s.} \\ 30 \text{ at } 6\text{d.} = 15\text{s.} \\ 30 \text{ at } 1\text{d.} = 2\text{s. } 6\text{d.} \\ 30 \text{ at } \frac{1}{2}\text{d.} = \underline{1\text{s. } 3\text{d.}} \\ \qquad\qquad 48\text{s. } 9\text{d.} \end{array}\right\} \text{or total cost} = 48\text{s. } 9\text{d.}$$

PROPORTION AND RATIO.

Meaning of Ratio.—A comparison of the *relative size* of two objects, without reference to their *absolute size*, gives the idea of proportion. When two quantities of the *same kind* are considered, the relation which one quantity bears to another is called a **ratio.** Such a comparison is made by considering how many times one quantity is contained in the other; thus the ratio of 12 things to 6 similar things is definite, and indicates that the number of one kind is twice that of the other; but the ratio of 12 tables to 6 chairs conveys no meaning. In comparing two quantities of the same kind, we can assert that one is twice, three times, or some multiple or sub-multiple of the other, without defining what the unit implies. The magnitudes may be either abstract or concrete numbers, but the ratio between them must always be abstract. Hence, it is necessary in comparing, that the quantities be written in terms of a common unit, as, for example, the ratio of 3 tons to 14 lbs., or the ratio of 10 feet to 4 inches; as there are 2240 lbs. in a ton, the first named ratio would be $3 \times 2240 : 14$; the second, $10 \times 12 : 4$.

Proportion.—The two ratios $2 : 4$ and $8 : 16$ are obviously equal, and their equality is expressed either by $2 : 4 = 8 : 16$, or by $2 : 4 :: 8 : 16$. When, as in the given example, the two ratios are equal, the four terms are said to be in *proportion*, hence:

Four quantities are said to be proportional, when the ratio of the first to the second is equal to the ratio of the third to the fourth. That is, when the first is the same multiple or sub-multiple of the second, which the third is of the fourth.

We may thus state that the numbers 6, 8, 15, and 20 form a proportion. The proportion is written as $6 : 8 :: 15 : 20$, and this should be read as 6 is to 8 as 15 is to 20.

The first and last terms of a proportion are called the **extremes**, and the second and third terms the **means**; thus 6 and 20 are the extremes, and 8 and 15 are the means in the above proportion.

When four quantities are proportional, the product of the extremes is equal to the product of the means. Thus $6 \times 20 = 8 \times 15$, or $\frac{6}{8} = \frac{15}{20}$, in which the proportion is written as the equality of two ratios.

Since the product of two of the terms of a proportion is equal to the product of the other two, it follows at once, that, if three terms of a proportion are given, the remaining one can be calculated.

Ex. 1. Find the second term of a proportion in which 14, 12, and 15 are respectively the 1st, 3rd, and 4th terms.

$$14 : \text{required term} :: 12 : 15;$$

$$\therefore \quad \text{required term} = \frac{15 \times 14}{12} = 17\tfrac{1}{2}.$$

Direct Proportion.—Three quantities are said to be in direct proportion when the first is to the second as the second is to the third.

Thus, $4 : 6 :: 6 : 9$ is a direct proportion.

$$\therefore \quad 4 \times 9 = 6^2, \text{ or } 6 = \sqrt{36}.$$

Expressed generally, if a, b, and c be in direct proportion, then:

$$a : b :: b : c, \quad \therefore \quad ac = b^2, \text{ or } b = \sqrt{ac}.$$

Mean Proportional.—In the above examples 6 is said to be a mean proportional to 4 and 9, and b is said to be a mean proportional to a and c.

Geometrical and Arithmetical Means.—The geometrical mean (written G.M.) between two numbers is found by taking the square root of their product. Thus, the G.M. of 4 and 9 is $\sqrt{4 \times 9} = 6$; and the G.M. of a and c is $\sqrt{ac}$.

The *arithmetical mean* (A.M.) is half the sum of the numbers.

The arithmetical mean of 4 and 9 is $\frac{4+9}{2} = 6{\cdot}5$.

The arithmetical mean of a and c is $\frac{a+c}{2}$.

In finding the ratio of one quantity to another, it is only the relative magnitudes of the two quantities which are of importance; the quantities themselves may be as small as possible, but the ratio of two very small quantities may be a comparatively large number.

Thus $\frac{1}{1000} = {\cdot}001$ is a small quantity, and so is ·00001, but the ratio of ·001 to ·00001 is $\frac{{\cdot}001}{{\cdot}00001} = 1000$. This very important fact concerning ratio is often lost sight of in calculations by beginners, and it must be carefully noted.

Percentages.—The rate of increase or diminution of one quantity as compared with another of the same kind is usually expressed in the form of a percentage. This enables a comparison to be made at once, without any preparatory labour of reducing fractions to like denominators. Examples on percentages occur so frequently, and are so varied, that it is difficult to select typical illustrations. The following, however, may make the matter clear.

Ex. 1. Suppose that two classes of 20 and 50 students respectively are expected to attend an examination. In the first named 18 students, and in the second 47 students, present themselves. Then we might say that 2 in 20 and 3 in 50 were away from the examination; but the comparison is most easily made by finding the percentage in each case. Thus in the first case we have

$$\tfrac{2}{20} \times 100 = 10 \text{ per cent.};$$

in the second case $\frac{3}{50} \times 100 = 6$ per cent.

These would be written as 10% and 6%.

Ex. 2. Suppose the population of a town in 1885 was 15,990, and in 1890 was 20,550. The actual increase $= 20550 - 15990 = 4560$; and although the actual increase is useful it is much better to be able to state the rate at which the population is increasing for each 100 of its inhabitants. The increase for each 100 of its population is found by simple proportion as follows:

$$15990 : 100 :: 4560 : \text{answer required.}$$

$$\therefore \quad \text{Answer} = \frac{4560 \times 100}{15990} = 28\dot{}5.$$

Thus, the increase for each 100 of its population is 28·5. This number is called 28·5 per cent., and is written 28·5%. The rate per cent. enables an increase or diminution to be referred to readily.

The population of another town in 1885 was 20,400, and in 1890 was 24,960. The actual increase (as before) is 4560, but it does not follow from this that the two towns are increasing at the same rate. As before the rate of increase is obtained from:

$$20400 : 100 :: 4560 : \text{answer.}$$

$$\therefore \quad \text{Answer} = \frac{4560 \times 100}{20400} = 22 \cdot 3.$$

Hence, the population of the latter is not increasing as fast as the former town by 6 per hundred, or by 6%.

In like manner percentages are often used to compare the proportions of lunatics, paupers, criminals, etc., in different towns, and rate or debt collectors and others are usually paid at the rate of so much per cent., *i.e.* if a rate collector is paid at the rate of 2 per cent., this would mean, that for every £100 collected he is allowed £2; for every £50, £1, etc.

Ex. 3. If, in a machine, it is found that a quarter of the energy expended is lost in frictional and other resistances, we should say that 25 per cent. is lost, meaning that $\frac{25}{100}$ is lost. This does not tell us the actual numerical amount of the loss, all that we can infer is, that for every 100 units of work expended on the machine 25 units disappear. It also enables a comparison to be made and is a convenient method of expressing the efficiency of machines. If one machine has an efficiency of 75 per cent. and another of 80 per cent., we know that the second is 5 per cent. more efficient than the first.

If, in addition, we know that 25 per cent. is the total loss due to all resistances, but 10 per cent. of this is due to the resistance of a particular part of the mechanism, this gives a *percentage of a percentage* and its numerical value is

$$\tfrac{25}{100} \text{ of } \tfrac{10}{100} = \tfrac{25}{100} \times \tfrac{10}{100} = \tfrac{250}{10000} = 2{\cdot}5, \text{ or } 2\tfrac{1}{2} \text{ per cent.}$$

Ex. 4. A reef of quartz contains ·0044 per cent. of gold. If the quartz produces £5. 12*s.* per ton, find the weight of a sovereign in grains.

$$£5.\ 12\text{s.} = £5\tfrac{6}{10} = 5{\cdot}6 \text{ sovereigns,}$$

$$1 \text{ ton} = 2240 \times 7000 \text{ grains,}$$

and $$\cdot0044 \text{ per cent.} = \frac{\cdot0044}{100} = \cdot000044.$$

$$\therefore \text{ weight of } 5{\cdot}6 \text{ sovereigns} = \cdot000044 \times 2240 \times 7000$$

$$= \cdot44 \times 224 \times 7.$$

$$\therefore \text{ weight of 1 sovereign} = \frac{\cdot44 \times 224 \times 7}{5{\cdot}6}$$

$$= 123{\cdot}2 \text{ grains.}$$

Ex. 5. Seventy-five per cent. of the area of a farm is arable; of the remainder eighty-five per cent. is pasture, and the rest is waste; the area of the waste is 3 ac. 0 r. 20 p. What is the area of the farm?

75 per cent. $= \tfrac{75}{100}$, or $\tfrac{3}{4}$ of the area is arable;

25 per cent. $= \tfrac{25}{100}$, or $\tfrac{1}{4}$ of the area is pasture and waste;

and of this quarter 85 per cent. is pasture, $\therefore$ 15 per cent. is waste.

But $\tfrac{15}{100} \times \tfrac{1}{4}$ of the area = 3 ac. 0 r. 20 p.

$\therefore$ $\tfrac{3}{80}$ of the area = 3 ac. 0 r. 20 p.

$\therefore$ area of the whole farm $= \tfrac{80}{3}$(3 ac. 0 r. 20 p.) = 83 ac. 1 r. $13\tfrac{1}{3}$ p.

EXERCISES. XVII.

1. A shopkeeper marks his goods with a price from which he can deduct $7\tfrac{1}{2}$ per cent. for prompt payment, and still have a profit of 10 per cent. on what the goods cost him: find the the cost price of an article which he marks at £2. 15s.

2. A quantity of ore containing 23 per cent. of copper is bought at 9s. per cwt.; 95 per cent. of the copper is extracted at a cost of 2s. $10\tfrac{1}{2}$d. per cwt. of ore: find the price per ton at which the copper must be sold if a profit of 15 per cent. is to be made.

3. I sell a house at a profit of 12 per cent. on what it cost me. Had I received £20 more for the house my profit would have been $14\tfrac{2}{7}$ per cent.: what did I give for the house?

4. The value of a certain house in 1880 has increased by 35 per cent. since 1877. The house was rated in 1877 at two-thirds of its value, and in 1880 it is rated at three-fifths of its value, the rate in the £ remaining the same. Compare the rate paid in 1877 with that paid in 1880.

5. A merchant buys certain goods at £35 per ton. At what price per lb. must he sell them in order to gain 10 per cent. on his outlay?

6. By selling 10 acres of land for £4699. 8s. 3d. a man gained $5\frac{1}{4}$ per cent.: whatwas the original price per acre?

7. A tradesman forms a mixture of tea by adding 1 lb. at 3s. and $\frac{1}{2}$ lb. at 4s. 6d. to every 2 lb. at 2s.: what must he sell it at per lb. in order to gain 10 per cent.?

8. If the annual increase in the population of a state is 25 per thousand, and the present number of inhabitants is 2,624,000: what will the population be in 3 year's time? And what was it a year ago?

9. If a man buys eggs at 10 for a shilling, and sells them at 8 for a shilling, what rate per cent. profit does he make on his outlay?

10. If 4 per cent. be lost by selling silk at 10s. per yard, at what price per yard should it be sold in order to gain 5 per cent.?

11. One gallon of spirit which contains 11 per cent. of water is added to three gallons containing 7 per cent. of water, and to this mixture half a gallon of water is added: find the percentage of water in the mixture.

12. If a grocer gains 10 per cent. by selling tea at 2s. 3d. per lb., what will he gain per cent. by selling it at 2s. 9d.?

13. Define ratio. Does it follow from your definition that it would be wrong to speak of the ratio of 5 tons to 3 miles, and, if so how does it follow?

Summary.

Involution.—The continued multiplication of a number by itself is called Involution. The number is called the root or first power, the second power is called the square, the third the cube. Thus $2^2=$ square of 2, $2^3=$cube of 2, etc.

Evolution.—Given a power of any number the process of finding the root is called Evolution. The sign $\sqrt{\ }$ indicates the square root. Thus $\sqrt{9}$ means the square root of 9. This is also written $9^{\frac{1}{2}}$. Similarly the cube root of nine is written $\sqrt[3]{9}$ or $9^{\frac{1}{3}}$.

Cube Root.—If a number is multiplied three times by itself the product is called the cube of the number, and the number itself is called the cube root of the product. Thus $3^3=27$ and $\sqrt[3]{27}$, or $27^{\frac{1}{3}}=3$.

Ratio.—A comparison of one number with another, obtained by dividing one by the other, is called Ratio. Hence the ratio of 3 to $9=\frac{3}{9}=\frac{1}{3}$.

Proportion is the equality of two ratios. The four numbers 3, 4, 15, and 20 form a proportion, or the ratio of the first two is equal to the ratio of the last two terms,

$$\therefore\ \frac{3}{4}=\frac{15}{20},\ \text{or}\ 3:4::15:20.$$

When three terms are known the fourth can be found. The product of the means, or, in the above example, 4 × 15, is equal to the product of the extremes, or 3 × 20.

CHAPTER VI.

ALGEBRA.

ADDITION, SUBTRACTION, MULTIPLICATION, AND DIVISION. SIMPLE OPERATIONS. FACTORS. SIMPLIFICATION OF ALGEBRAICAL EXPRESSIONS. INDICES.

Explanation of Symbols.—In operating with numbers or digits as the numerals 1, 2, 3 ... are called, accurate results are obtained whatever be the unit employed. Thus, the digit 7 may refer to 7 shillings, ounces, yards, or other units. In adding two digits, such as 7 and 5, together—taking care that the quantities are of the same kind—we obtain the sum 12, whatever the unit employed may be.

The signs already made use of in Arithmetic are also employed in Algebra, but in Algebra representations of quantities are utilised which have a further generality. Both letters and figures are used as symbols for numbers or quantities. These numbers may be *known* numbers and are then usually represented by the first letters of the alphabet a, b, c, etc., or they may be numbers which have to be found, called *unknown* numbers, and these are often denoted by x, y, z.

A more general meaning is given to the signs + and −.

If a distance AC measured along a line is said to be positive, the distance CA measured in the opposite direction would be negative.

The result of the first measurement would be indicated by $+\mathbf{a}$, while the same distance CA, but *measured in the opposite direction*, would be indicated by $-\mathbf{a}$.

Again, if a length AB be measured in the same direction from left to right and be denoted by $+\mathbf{b}$, the length BA measured from right to left would be indicated by $-\mathbf{b}$.

Hence, $+a+b$ would mean the sum or addition of the two lines and so a line of length equal to AD is obtained, where

$$AD=AC+AB.$$

Similarly, $+a-b$ would denote the length BC obtained by measuring a length a in the positive and a length b in the negative direction.

When writing down an expression it is usual, where possible, to place the positive quantity first and to dispense with the + sign. The above expressions would, therefore, always be written as $a+b$ and $a-b$. The signs placed between the numbers indicating in the first case the sum of two positive quantities, and in the second case the subtraction of one positive quantity from another; in the latter case, the quantity $a-b$ could also be described as the addition of a negative quantity b to a positive quantity a, this is called the **algebraical sum** of the two quantities, or the algebraical sum of two quantities is the result after carrying out the operation indicated by the signs before the two quantities.

The algebraical sum of $+10$ and $-18=-8$.

In the quantity $a-b$, if a represents a sum of money received, then $-b$ will represent a sum of money paid away. The algebraic sum is represented by the balance $a-b$.

Again, a may represent the height of a point P *above* a certain level AB, then $-b$ may represent the distance of another point D below the same level.

It will be seen that in Algebra the word sum is used in a different and a wider sense than in Arithmetic. Thus, in Arithmetic $a-b$ indicates that b is to be subtracted from a, but in Algebra it also means the sum of the two quantities.

The arithmetical symbols of operation, +, −, ×, and ÷, are used in Algebra, but are varied according to circumstances; the general sign for the multiplication of quantities is × ; but the product of single letters may be expressed by placing the letters one after another, thus the product of a and b may be written $a\times b$ but is usually written as ab. In a similar manner the product of $4a$, x, and y is expressed by $4axy$.

The product of two quantities $a+b$, and $c+d$ may be expressed as $(a+b)\times(c+d)$, or usually as $(a+b)(c+d)$.

Multiples of the quantities a, b, c, etc., may be expressed by placing numbers before them as, $2a$, $3b$, $5x$; the numbers 2, 3, and 5 thus prefixed are called the coefficients of a, b, and x.

The product of a quantity multiplied by itself any number of times is called a **power** of that quantity and is indicated by writing the number of factors on the right of the quantity and above it. Thus:

$a\times a$ is called the square of a and is written a^2;

$b\times b\times b$ is called the cube of b and is written b^3.

Similarly, $c\times c\times c \ldots n$ factors is written c^n and indicates c to the power n.

The number denoting the power of a given quantity is called its **index** or **exponent.**

It is very important that the distinction between coefficient and index be clearly understood. Thus $4a$ and a^4 are quite different terms.

Ex. 1. Let $a=2$, then $4a=8$; but $a^4=2^4=16$.

The use of signs may be exemplified in the following manner:

Ex. 2. In the expression a^2+b-c.

Let $a=4,\ b=7,$ and $c=3$.

Then $a^2+b-c=4^2+7-3=23-3=20$.

Ex. 3. Find the value of

$$\frac{ax^2+b^2}{bx-a^2-c}, \text{ when } a=3,\ b=5,\ c=2,\ x=6.$$

Here $ax^2+b^2=\ 3\times6^2+5^2\ =133,$

also $bx-a^2-c=5\times6-3^2-2=\ 19,$

$$\frac{ax^2+b^2}{bx-a^2-c}=\frac{133}{19}=7.$$

Ex. 4. Find the value of

$$(ac-bd)\sqrt{a^2bc+b^2cd+c^2ad-2},$$

when $a=-1$, $b=2$, $c=3$, $d=0$.

Substituting these values in the given equation we obtain

$$(-3-0)\sqrt{1\times2\times3+4\times3\times0+(9\times\ -1\times0)-2}$$
$$=-3\sqrt{6-2}=-6.$$

Ex. 5. **Find the numerical value of**

$$\frac{x^2}{a^2}+\left(\frac{z-x}{b}\right)^2, \text{ when } x=3,\ z=4,\ a=b=5.$$

$$\frac{x^2}{a^2}+\left(\frac{z-x}{b}\right)^2=\frac{3^2}{5^2}+\left(\frac{4-3}{5}\right)^2=\frac{9}{25}+\frac{1}{25}$$

$$=\frac{10}{25}=\frac{2}{5}$$

Ex. 6. **Find the value of**

$$(b-c)^3+2(c-a)^3+(a-b)^3-3(b-c)(c-a)(a-b),$$

when $a=1,\ b=-\frac{1}{2},\ c=\frac{3}{2}.$

Substituting the given values, we have

$$\left(-\frac{1}{2}-\frac{3}{2}\right)^3+2\left(\frac{3}{2}-1\right)^3+\left(1+\frac{1}{2}\right)^3-3\left(-\frac{1}{2}-\frac{3}{2}\right)\left(\frac{3}{2}-1\right)\left(1+\frac{1}{2}\right)$$

$$=-8+\frac{1}{4}+\frac{27}{8}+\frac{9}{2}=-8+\frac{65}{8}=\frac{1}{8}.$$

EXERCISES. XVIII.

Find the value of

1. $\dfrac{3a+4b}{3x-a-b}-\dfrac{a^2-2b^2}{x^2-b^2}$, **when** $a=10,\ b=5,\ x=7.$

2. $6\sqrt{ax^2}-\dfrac{2ax^2}{(a-x)^2}-a^2$, **when** $a=4,\ x=2.$

3. $3xy-\dfrac{3(x-y)}{x+y}-4\sqrt{2xy}$, **when** $x=4,\ y=2.$

4. $\dfrac{a^2+x^2}{a^2-x^2}+\dfrac{5\sqrt{ax}}{a+x}-\dfrac{(a-x)^2-11}{5}$, **when** $a=9,\ x=4.$

5. $\dfrac{x^2+y^2-z^2}{yz+zx-xy}-\dfrac{\sqrt{x+y+z}}{2(y+z-x)}$, **when** $x=4,\ y=3,\ z=2.$

6. $\dfrac{x^3-a^3}{x-a}$, **when** $x=4,\ a=-2.$

7. $\dfrac{a^3-b^3}{a-b}$, **when** $a=3,\ b=2.$

If $a=7,\ b=5,\ c=4,\ d=2$, **find the value of**

8. $3a-2b+4c.$
9. $4(a-b)(4a-bc).$
10. $10a^2-4abc+3b^2.$
11. $(a^2-b^2)(a-b).$
12. $(a+b-c)(3a-b-2c).$
13. $\dfrac{(a-b)^2}{b-c}+(a-c)^2.$

Find the numerical values of the following expressions :

14. $\dfrac{\sqrt{a+b}}{\sqrt{3a}-b}-\dfrac{a+10b}{a-b}$, when $a=5$, $b=-1$.

15. $\dfrac{a+x}{a-x}+\dfrac{\sqrt{a+x}}{\sqrt{a+2x}}$, when $a=7$, $x=-16$.

16. (*a*) $\sqrt{(8y+2z+7)}+\sqrt{(6x-8y+z)}$; (*b*) $\dfrac{x}{y}+\dfrac{4x+2z}{2x-z}$;
(*c*) $xy+yz-zx$; when $2x=3$, $4y=3$, and $z=-2$.

17. (*a*) $6xy-4yz+2zx$; (*b*) $\dfrac{9x}{8y^2}+\dfrac{24y}{z^2}+\dfrac{z}{36x^2}$; (*c*) $\dfrac{3x}{y+z}+\dfrac{z}{x+y}$;
when $3x=1$, $2y=-3$, $z=-4$.

18. (*a*) $\dfrac{x}{3y}-\dfrac{y}{2z}$; (*b*) $x^2y^3+2y^2z-3yz^2$; (*c*) $\dfrac{1}{x+y}+\dfrac{2}{y+z}$;
when $2x=1$, $3y=-4$, $2z=7$.

To show that the subtraction of a negative quantity is equivalent to adding a corresponding positive quantity.

If a length AB be denoted by a and another BC by b, then $a+b$ would be represented by a line $AB+BC$ or by AC, both being measured in the positive direction (from left to right).

Also $a-b$ would be a quantity obtained by subtracting b from a, and could be obtained by measuring off a length BD in a negative direction, so that $a-b$ is appropriately represented by AD

As BC is positive, the reversal of direction indicated by CB is negative, and would be indicated by $-b$. Now, to subtract b from a, we reversed the direction of b and added it on to a. If then we have to subtract a negative quantity, $-b$ or CB, from a positive quantity a, reversing the direction we obtain BC or $+b$, and adding on to a we get AC or $a+b$. We could indicate this by $a-(-b)$, the negative sign outside the bracket indicating that the quantity inside the bracket has to be subtracted from a. The change in sign is true whether the quantity subtracted be positive or negative. Hence, to subtract one quantity from another—change the sign of the quantity subtracted and proceed to add the two together.

Use of Brackets.—In Algebra it is frequently necessary to enclose parts of an expression, and the use of brackets for this purpose is very important. There are several forms of brackets in general use. These are ( ), { }, [ ]. Sometimes a line is placed over two numbers with the same meaning as enclosing in brackets. Thus, some quantity $b+c$ which is to be multiplied by $d+f$ may be written as $\overline{b+c}\times\overline{d+f}$, or as $(b+c)(d+f)$. The use of the different forms of brackets will be understood by the following examples :

Ex. 1. $3a-(4b-7c)$.

Here the brackets indicate that $4b-7c$ is to be subtracted from $3a$, and it is obvious that the result obtained will be the same whether we subtract $7c$ from $4b$, and afterwards subtract the remainder from $3a$, or add $7c$ to $3a$, and subtract $4b$ from the sum.

A positive or negative sign may be placed before a bracket: if the former, then the signs of all the terms are unaltered when the brackets are removed ; if the latter (or negative sign), the signs of all the terms inside the brackets are altered.

Ex. 2. $3a+(4b-7c+3d)=3a+4b-7c+3d.$
$3a-(4b-7c+3d)=3a-4b+7c-3d.$

The other forms of brackets which are used are [ ] and { }. In each case they denote that whatever is in one pair of them is to be regarded as one quantity to be added, subtracted, multiplied or divided as the signs and quantities outside the brackets indicate.

Ex. 3. Express the product of $2a+3b$ and $4c+5d$.

The quantities must be written as $(2a+3b)(4c+5d)$.

Further, to indicate that $3f$ is to be subtracted from the product and the result multiplied by $7e$, we use another pair of brackets, thus, $7e\{(2a+3b)(4c+5d)-3f\}$; and to express that when $3x$ is subtracted from the last obtained product the whole must be multiplied by 8, we have to use still another bracket, thus,
$$8[7e\{(2a+3b)(4c+5d)-3f\}-3x].$$

Conversely, the brackets are removed in the reverse order, *i.e.* beginning with [ ].

EXERCISES. XIX.

Simplify

1. $2[4x-\{2y+(2x-y)-(x+y)\}]$.
2. $5x-(2y-x-1)-2\{3y-2(x+y)\}$.
3. $3x-4y-2(x-2y)+\frac{3}{2}\{2x-1-3(x-y)\}$.

4. Reduce $\{x^2-(x-a)^2\}\times\{(x+a)^2-x^2\}\times\{(2x^2+a^2)^2-4x^4\}$ to its equivalent form without brackets.

5. Simplify

$$2x-[a-(3b-5c)+\{x-(4a-b+x-c)+3c\}-(2b-3a+3c)].$$

6. Add together the quantities:

$$x-[1-(b-c)-a],\ x-[1-(c-a)-b],\ x-[1-(a-b)-c],$$
$$4-(a+b+c+3x).$$

7. Divide $x^4-2bx^3-(a^2-b^2)x^2+2a^2bx-a^2b^2$ by $x^2+(a-b)x-ab$.

8. Find the value of

$$\frac{4y}{5}(y-x)-35\left[\frac{3x-4y}{5}-\frac{1}{10}\left\{3x-\frac{5}{7}(7x-4y)\right\}\right],$$

when $x=-\frac{1}{2}$ and $y=2$.

9. Show directly from the meanings of the symbols employed that

$$a+(b-c)=a+b-c,$$

and

$$a-(b-c)=a-b+c.$$

10. Reduce to the simplest possible form:

$$4\left\{a-\frac{3}{2}\left(b-\frac{4c}{3}\right)\right\}\left\{\frac{1}{2}(2a-b)+2(b-c)\right\}.$$

11. Simplify the expression:

$$a-\{3a+c-(4a-[3b-c])+3b\}.$$

12. Simplify the expression:

$$\frac{1}{4}\left\{a-5(b-a)\right\}-\frac{3}{2}\left\{\frac{1}{3}\left(b-\frac{a}{3}\right)-\frac{2}{9}\left[a-\frac{3}{4}\left(b-\frac{4a}{5}\right)\right]\right\},$$

and find the value of

$$\{a-(b-c)\}^2+\{b-(c-a)\}^2+\{c-(a-b)\}^2$$

when $a=1$, $b=3$, $c=5$.

Addition.—The addition of algebraical quantities denotes the putting down in one sum all the like quantities, regard being had to their signs.

When like quantities have the same sign, their sum is found by adding the coefficients and annexing the common letters. Thus $7a+4a=11a$. Also, $7a+3a+3b+5a=15a+3b$.

When several quantities have to be added together, they may be written in columns; and in adding the positive and negative coefficients, add separately the sign of the greater and annex the common letters.

The operation would proceed as follows:

7 and 3 are 10; subtract 5 and the sum is $5a$, as shown. In a similar manner $7b$ is obtained. Hence, the sum required is $5a+7b$.

$$\begin{array}{r}7a+5b\\-5a+4b\\3a-2b\\\hline 5a+7b\\\hline\end{array}$$

EXERCISES. XX.

Add together

1. $a-3b+3c-d$ and $a+3b+3c+d$.
2. $4x^2-3x+4$, $-2x^2+x-5$, $3x^2-5x+1$, $7x^2+2x-4$, $-x^2-4x+13$.
3. $-7ab+3bc-xy$, $-ab+2bc+4xy$, $3ab-bc+2xy$, $-2ab+4bc-3xy$, $5ab-8bc+xy$.
4. $3c+5x^2$, $-2c-x^2$, $c+x^3$, $-5c+5x^2$, $c-10x^2$.
5. $7xy-4az+2bc$, $6az+5mn-3p$, $2bc-3xy+8az-xy-bc-az$, $-4az+3xy-4bc$.
6. $-2yz-3ab$, $-5cx+2yz+5ab$, $7ab-3cx+yz$, $4ek-6yz-cx$, $2cx+yz-8$.
7. Add together $a-b+c$, $2a+2b-3c$, $-3a+4b+4c$; and subtract the sum from $4a+5b-5c$.

Add together the following:

8. $2a^2+ab-3b^2$, $a^2-2ab+4b^2$, $-4a^2+5ab-6b^2$.
9. $3a^3-5a^2b+2b^3$, $8a^2b-3b^3+2ab^2$, $5ab^2-4a^3-3a^2b$, $2a^3-6ab^2+4b^3$.

Subtraction.—In Algebra, to perform the operation of subtraction, change the signs of all the terms to be subtracted, and then add to the other expression. Thus, to subtract $7a$ from $13a$, we reverse the sign of $7a$ and make it minus; for $13a-7a$ is only another way of expressing that $7a$ is to be subtracted from $13a$.

Ex. 1. From $5a+3x-2b$ subtract $2c-4y$. The quantity to be subtracted when its signs are changed is $-2c+4y$.

$\therefore$ the remainder is $5a+3x-2b-2c+4y$;

Ex. 2. Subtract $a^2-2b-2c$ from $3a^2-4b+6c$.

Here, arranging as in addition and changing the sign, we proceed as in addition, thus:

$$\begin{array}{r} 3a^2-4b+6c \\ -\ a^2+2b+2c \\ \hline 2a^2-2b+8c \end{array}$$

$3a^2-a^2=2a^2$, etc.

EXERCISES. XXI.

1. Subtract $a+2b-c$ from $4a-3b+c$.
2. $4z+y^2+3x-20$ from $7z+5y+x+10$.
3. $4x^3-3x^2-x+2$ from $7x^3-6x^2+2x-1$, and find the value of the answer when $x=3$.
4. $y(6az+7cx)-4y(bx-cz)$ from $2(3a+2c)yz-3(b-2c)xy$.

5. Subtract $-2mz-3bx+4ay-3$ from $8mz+bx-6ay-2$.

6. Subtract (i) $3x^3+4x^2y-7xy^2+10y^3$ from $4x^3-2x^2y+4xy^2+4y^2$; and (ii) simplify $7a-2[3a-2b+\{(a+b)-(a-b)\}]$.

7. Subtract $2a-4b+5c-6d$ from $3a+6b-8c-d$.

8. Subtract $2a-3\{a-(b-a)\}$ from $2b-3\{b-(a-b)\}$

9. Subtract $4x^3-3x^2-x+2$ from $7x^3-6x^2+2x-1$,

10. From $2(3a+2c)yz-3(b-2c)xy$ subtract $y(6az+7cx)-4y(bx-cz)$.

11. Subtract $3\{x+2y-\frac{2}{3}(y-4)\}-4(3x-y+2)$ from $3x+2-[7y-4x-\{4(x-2)-3(2y+1)\}]$.

12. Subtract $(x^2-7x+13)^2$ from $(x^2+7x-13)^2$.

13. Subtract $6x+1-2(x+5y)-\{2-(x+2y-1)\}$ from $2(2-y)-7x-2\{1-3y+6(y-x)\}$.

Multiplication.—As already seen in Arithmetic, multiplication may be considered as a concise method of finding the sum of any quantity when repeated any number of times; the sum thus obtained is called the *product*; in Arithmetic, the sign × is used, but in Algebra, this may, or may not be used; or, $ab=a\times b$; $a(x-y)=a\times(x-y)$, etc.

In multiplying, what is called the Rule of Signs must be observed, *i.e. The product of two terms with like signs is positive; the product of two terms with unlike signs is negative.*

As this usually presents some difficulty, we may, with advantage, explain the reason for the rule, thus:

If a is to be multiplied by b, it means that a has to be added to itself as often as there are units in b; hence the product is ab.

If $-a$ is to be multiplied by $-b$, it means that $-a$ is to be subtracted as often as there are units in b. But since to subtract a negative quantity is the same as to add a positive one (p. 80), the product is again ab.

Again, if $-a$ is to be multiplied by b, it means that $-a$ is to be added to itself as often as there are units in b, hence the product is $-ab$; obviously, the same result would be obtained by multiplying a by $-b$.

RULE. To multiply two simple expressions together, *multiply the coefficients and add the indices of like letters.*

Ex. 1. $4ab \times 3a^2b^2 = 12a^3b^3$.

Ex. 2. $4a^3b^2c^5e^4 \times 6a^4b^3c^4e^3 = 24a^7b^5c^9e^7$.

When the expressions consist each of two terms the process of multiplication is conveniently arranged as follows:

Ex. 3. Multiply $x+5$ by $x+6$.

$$\begin{array}{lll} x + 5 & & \\ x + 6 & & \\ \hline x^2 + 5x & & \\ \quad\;\; 6x + 30 & & \\ \hline x^2 + 11x + 30 & & \end{array}$$

Arrange the two expressions as above, one under the other; multiply each term of the first by each term of the second, and arrange as shown; finally, add the terms together to obtain the product. By arranging the terms in this way, one under the other, and multiplying, the result can always be obtained. But this is not enough; the student should be able to write down at once (by inspection) the product.

This is effected by noting that the first term x^2 of the product is obtained by multiplying together the two first terms in the given expressions; the last term is the product of the two second terms 6 and 5, and the middle term is the sum of the two given terms (taken with their proper signs), *i.e.* $x \times 5 + x \times 6$;

$$\therefore\ (x+5)(x+6) = x^2 + 11x + 30.$$

In a similar manner,

$$(a+b)^2, \text{ or } (a+b)(a+b) = a^2 + 2ab + b^2,$$
$$(a-b)^2, \text{ or } (a-b)(a-b) = a^2 - 2ab + b^2,$$
$$(x-5)(x-6) = x^2 - 11x + 30;\ (x-5)(x+6) = x^2 + x - 30;$$
$$(x+5)(x-6) = x^2 - x - 30.$$

When the product of two expressions containing more than two terms is required, it is usually convenient to arrange the terms one under the other, and proceed as in the following example.

Ex. 3. Multiply together $14ac - 3ab + 2$ and $ac - ab + 1$.

$$\begin{array}{llllll} 14ac & -3ab & +2 & & & \\ ac & -ab & +1 & & & \\ \hline 14a^2c^2 & -3a^2bc & +2ac & & & \\ & -14a^2bc & & +3a^2b^2 & -2ab & \\ & & +14ac & & -3ab & +2 \\ \hline 14a^2c^2 & -17a^2bc & +16ac & +3a^2b^2 & -5ab & +2 \end{array}$$

EXERCISES. XXII.

Multiply

1. x^3+6x^2+8x-8 by x^2-2x+4.
2. $6x^3+13x^2-4x-15$ by $6x^3-13x^2-4x+15$.
3. $2a+3b-4c$ by $2a-3b-4c$.
4. x^4+x^3-x-1 by x^2-x+1.
5. $4x^4+6x^3y+9x^2y^2$ by $4x^4-6x^3y+9x^2y^2$.
6. $x^2+y^2+z^2-xy-xz-yz$ by $x+y+z$.
7. $\frac{x^2}{y^2}+1+\frac{y^2}{x^2}$ by $\frac{x}{y}-\frac{y}{x}$.
8. $x^2+5x+\frac{25}{2}$ by $x^2-5x+\frac{25}{2}$.
9. $6x^3+13x^2-4x-15$ by $6x^3-13x^2-4x+15$.
10. x^4+x^3-x-1 by x^2-x+1.
11. $4x^4+6x^3y+9x^2y^2$ by $4x^4-6x^3y+9x^2y^2$.
12. $x^2+y^2+z^2-xy-xz-yz$ by $\frac{x}{2}+\frac{y}{2}+\frac{z}{2}$.
13. $x^4+4x^3+12x^2+16x+16$ by x^2-4x+4.
14. $a^3-3a^2x+3ax^2+x^3$ by $a^2+3ax-x^2$.

Division.—In Algebra, as in Arithmetic, the object of division is to determine how often one quantity called the **divisor** is contained in another called the **dividend**, or to find *a quantity* called the **quotient** by which *the divisor must be multiplied to produce the dividend.* Hence, division is the reverse of multiplication;

$$\therefore \text{ quotient}=\frac{\text{dividend}}{\text{divisor}}.$$

Ex. 1. Divide $18ax^2$ by $9ax$;

$$\therefore \frac{18ax^2}{9ax}=2x.$$

Ex. 2. Divide $15a^2b^2$ by $-5a$;

$$\therefore \frac{15a^2b^2}{-5a}=-3ab^2.$$

In each case multiplying divisor and quotient together we obtain the dividend.

When the dividend and divisor both consist of several terms, arrange both dividend and divisor according to the powers of the same letter, beginning with the highest. The following example worked out in full will show the method adopted:

$$\begin{array}{l}
a^2+2ax+x^2\,)\,a^5+5a^4x+10a^3x^2+10a^2x^3+5ax^4+x^5\,(\,a^3+3a^2x+3ax^2+x^3\\
\qquad\qquad\quad \underline{a^5+2a^4x+a^3x^2\qquad}\\
\qquad\qquad\qquad\quad 3a^4x+9a^3x^2+10a^2x^3\\
\qquad\qquad\qquad\quad \underline{3a^4x+6a^3x^2+3a^2x^3\qquad}\\
\qquad\qquad\qquad\qquad\qquad 3a^3x^2+7a^2x^3+5ax^4\\
\qquad\qquad\qquad\qquad\qquad \underline{3a^3x^2+6a^2x^3+3ax^4\qquad}\\
\qquad\qquad\qquad\qquad\qquad\qquad\quad a^2x^3+2ax^4+x^5\\
\qquad\qquad\qquad\qquad\qquad\qquad\quad \underline{\underline{a^2x^3+2ax^4+x^5}}
\end{array}$$

EXERCISES. XXIII.

Divide

1. $35x^4-22x^3+58x-35$ by $5x^2-6x+7$.
2. $x^4-9x^2-6xy-y^2$ by x^2+3x+y.
3. $3x^4+14x^3+9x+2$ by x^2+5x+1.
4. $6a^4+4a^3x-9a^2x^2-3ax^3+2x^4$ by $2a^2+2ax-x^2$.
5. $64x^6-48x^4y^2+12x^2y^4-y^6$ by $8x^3-12x^2y+6xy^2-y^3$.
6. $a^5-x^5-5ax(a^3-x^3)+10a^2x^2(a-x)$ by $a^2-2ax+x^2$.
7. $x^3+3x^2a+3xa^2+a^3+b^3$ by $x+a+b$.
8. $35x^4-22x^3+58x-35$ by $5x^2-6x+7$.
9. $x^4-9x^2-6xy-y^2$ by x^2+3x+y.
10. $64x^6-48x^4y^2+12x^2y^4-y^6$ by $8x^3-12x^2y+6xy^2-y^3$.
11. $63a^2-115ab+54ac-6bc+12b^2$ by $36a-4b$.
12. $x^3+3x^2a+3xa^2+a^3+b^3$ by $x+a+b$.
13. $x^2+1+\dfrac{1}{x^2}$ by $x+1+\dfrac{1}{x}$.
14. (i.) $x^5+8x^4-26x^3+117x^2-220x+64$ by $x^2+11x-4$;
    (ii.) $x^3+y^3+z^3-3xyz$ by $x+y+z$.
15. $15x^5-17x^4-24x^3+138x^2-130x+63$ by $5x^3+6x^2-9x+7$, and verify your result by multiplication.

Factors.—When an algebraic expression is the product of two or more quantities, each of these quantities is called a **factor** of it. Thus if $x+5$ be multiplied by $x+6$, the product is $x^2+11x+30$, and the two quantities $x+5$ and $x+6$ are said to be the factors of $x^2+11x+30$. The determination of the factors of a given expression, or as it is called the **resolution** of the expression into its factors, may be regarded as the inverse process of multiplication. The following results easily obtained by multiplication, and which occur so frequently, are of the utmost importance, and should be carefully remembered:

$$(a+b)(a+b) \text{ or } (a+b)^2=a^2+2ab+b^2. \text{ ....}(1)$$

Also $$(a-b)^2=a^2-2ab+b^2. \text{ ....}(2)$$

The results are true when any other letters are used instead of a and b. Hence we can write with equal correctness

$$(x+y)^2=x^2+2xy+y^2.$$

Or, *The square of the sum of two quantities is equal to the sum of the squares of the quantities increased by twice their product.*

Similarly, *The square of the difference of two quantities is equal to the sum of the squares of the quantities diminished by twice their product.*

By multiplying $(x+y)(x-y)$ we obtain x^2-y^2. Conversely given x^2-y^2 we can at once write down the factors as $x+y$ and $x-y$. The first of these relations may be expressed as: *The product of the sum and the difference of two numbers is equal to the difference of their squares.*

Ex. 1. $40^2-39^2=(40+39)(40-39)=79\times 1=79$.

Ex. 2. To obtain the factors of x^4-y^4.

We first obtain $(x^4-y^4)=(x^2+y^2)(x^2-y^2)$.

Also as $$x^2-y^2=(x+y)(x-y),$$

we can write $$x^4-y^4=(x^2+y^2)(x+y)(x-y).$$

Multiplying a^2-ab+b^2 by $a+b$ the product is found to be

$$a^3+b^3.$$

$$a^3+b^3=(a+b)(a^2-ab+b^2)$$

Similarly $$a^3-b^3=(a-b)(a^2+ab+b^2).$$

The quantities $(a+b)(a^2-ab+b^2)$ are the **factors** of a^3+b^3, and $(a-b)(a^2+ab+b^2)$ are the factors of a^3-b^3.

Generally a^n+b^n is divisible by $a+b$ when n is an odd number, 1, 3, 5 etc. Thus in the above example n is 3.

Also a^n-b^n is divisible by $a-b$ when n, as before, is an odd number. The case of $n=3$ is shown, and by actual division, assuming n to be any odd number, the rule can be verified.

Also, as above explained, x and y or any two letters may be used instead of a and b.

When n is an even number, 2, 4, etc., it will be found that a^n-b^n is divisible by both $(a+b)$ and $(a-b)$.

Ex. 3. Let $n=6$; $\therefore\ a^6-b^6$.

We have $a^6-b^6=(a^3+b^3)(a^3-b^3)$,

and the factors are

$$(a+b)(a^2-ab+b^2)(a-b)(a^2+ab+b^2)$$

Hence a^6-b^6 is divisible by $a+b$ and $a-b$.

The case when $n=4$ has already been shown.

Hence a^n+b^n is divisible by $a+b$ when n is odd.
a^n-b^n „ „ $a-b$ „ „
a^n-b^n „ „ $a+b$ and $a-b$ when n is even.

As already indicated, it is possible with a little practice to write down the product by inspection of two terms of the form $(x+2)$ and $(x+3)$. Thus $(x+2)(x+3)=x^2+5x+6$.

To find the factors of x^2+5x+6,
the first term is the product of x and x;
„ last „ „ „ „ 2 „ 3;
„ middle „ „ sum „ 3 „ 2,
and in this manner the factors can be easily determined.

Ex. 4. Resolve into factors $x^2+8x+12$.

Here the two numbers required must have a sum of 8 and a product equal to 12. Of such pairs of numbers the sum of which is 8, are 4 and 4, 7 and 1, and 6 and 2, but only the last pair have a product 12. Hence the factors required are $(x+2)(x+6)$.

Or we could write the given expression

$$x^2+8x+12 \text{ as } x^2+2x+6x+12;$$

taking out a quantity common to two terms we obtain

$$x(x+2)+6(x+2).$$

This shows that $x+2$ is common to both terms, hence we may write

$$x^2+8x+12=(x+6)(x+2).$$

In a similar manner

$$\begin{aligned}x^2-9x+20&=x^2-5x-4x+20\\&=x(x-5)-4(x-5)=(x-4)(x-5).\end{aligned}$$

Ex. 5. $x^2+11x+30=x^2+5x+6x+30=(x+6)(x+5)$.

The factors in the preceding and other examples may also be found by substituting for x some quantity which will reduce the given expression to zero. Such a quantity or number is a factor. Thus in $x^2-9x+20$, the last term suggests that two of the following, 4 and 5, or 10 and 2, or 20 and 1, are the factors, but the middle term denoting the sum of the numbers gives -4 and -5.

To ascertain if 4 and 5 are the required factors put $x=4$.
$\therefore\ 16-36+20=0$. Hence $x=4$, or $x-4$ is a factor.
Similarly putting $x=5$ we obtain $25-45+20=0$;

$$\therefore\ x-5 \text{ is a factor.}$$

Hence $$x^2-9x+20=(x-4)(x-5).$$

Ex. 6. $x^2+6x-55$.
Put $x=-11$, this reduces the given expression to zero;

$$\therefore\ x+11 \text{ is a factor.}$$

Next put $x=+5$, and it is found to be a factor;

$$\therefore\ x^2+6x-55=(x+11)(x-5).$$

Ex. 7. $x^2+2x-3=(x+3)(x-1)$.

EXERCISES. XXIV.

Resolve into factors:

1. (i) x^3+27y^3; (ii) $x^2-13x+40$.

2. a^8-x^8 into four factors.

3. Show that $(x-1)^3$ is a common factor of

$$x^3(x^6+1)-3x^4(x^4-1)+3x^5(x^2-1)$$

and $$x^4(x^5+1)-3x^5(x^3+1)+2x^6(x+1);$$

and find the other common factors of these two expressions.

4. Write down the factors of

(i) $9x^2-64$; (ii) x^3+1; (iii) $(x+1)^4-1$.

5. Reduce to its lowest terms

$$\frac{2x^4+17x^3+30x^3+8x-5}{x^4+4x^3-18x^2-29x-10}.$$

6. Simplify (i) $\dfrac{6x^2-5xy-6y^2}{14x^2-23xy+3y^2}-\dfrac{15x^2+8xy-12y^2}{35x^2+47xy+6y^2}$;

(ii) $\dfrac{c}{ab(b-c)(c-a)}+\dfrac{b}{ac(a-b)(b-c)}+\dfrac{a}{bc(a-b)(c-a)}$.

7. Resolve into factors:

(i) $x^2+7x-78$; (ii) $xy+4x-9y-36$;
(iii) $(1+xy)^2-(x+y)^2$; (iv) $a^2-b^2-c^2+d^2+2(bc-ad)$.

8. Express $x^3+2x^2-9x-18$ in three factors.

9. Show that the following expression is the product of three factors: $3a(2a^2+b^2)+9a^2b+b(a+b)(2a+b)$.

10. Write down the following expressions in factors:
(i) $2x^2+5x-12$; (ii) $x^2+y^2-z^2+2xy$; (iii) $x^3+x^2y-xy^2-y^3$.

11. Multiply together x^2-3x+2, x^2-2x-3, and x^2+5x+6; and divide $7x^3-23x^2y+7xy^2-3y^3$ by $x-3y$.

12. Find by how much $(x^2-3x+1)^2$ is greater than

$$x(x-1)(x-2)(x-3)$$

Write down the following expressions in factors:

13. (i) x^2+x-2; (ii) $x^2-6xy+9y^2-z^2$; (iii) $2x^3+3ax^2-2a^2x-3a^3$.

14. (i) $x^2+7x-30$; (ii) $10x^3-40xy^2$; (iii) $x^3+x^2y+xy^2+y^3$.

15. x^2-x-2, $x^4+4x^2y^2+16y^4$, $x^2-y^2-6x+6y$.

16. (i) $\frac{a^2}{16}-\frac{3}{2}\cdot ab+9b^2$; (ii) $(x+y)^2-3(x+y)+2$;

(iii) $x^2+5x+4-y^2+3y$.

Indices.—As already explained (p. 62), the number which expresses the power of a quantity is called the **index**. Thus in a^5, a^7, a^9 the numbers 5, 7, and 9 are called the **indices** of a, and are read "a to the power five," "a to the power seven," etc.

Since $a^3\times a^2=(a\times a\times a)(a\times a)=a^{3+2}=a^5$, hence *to multiply together different powers of the same quantity it is only necessary to add the index of one to that of the other.*

This may be written in a more general manner as follows:

$$a^m=(a\times a\times a\ldots \text{ to } m \text{ factors}),$$

when m is a positive integer.

$$a^n=(a\times a\times a\ldots \text{ to } n \text{ factors});$$

$$\therefore\ a^m\times a^n=(a\times a\times a\ldots \text{ to } m \text{ factors})(a\times a\times a\ldots \text{ to } n \text{ factors})$$
$$=(a\times a\times a\ldots \text{ to } m+n \text{ factors})=a^{m+n}.$$

This most important rule has been shown to be true when $m=3$ and $n=2$. Other values of m and n should be assumed, and a verification obtained.

Also $$\frac{a^5}{a^3}=\frac{a\times a\times a\times a\times a}{a\times a\times a}=a^{5-3}=a^2.$$

Similarly $$\frac{a^m}{a^n}=\frac{a\times a\times a\ldots \text{ to } m \text{ factors}}{a\times a\times a\ldots \text{ to } n \text{ factors}}=a^{m-n}.$$

It is found convenient to use both fractional and negative indices.

Thus $\sqrt{a}$ is written as $a^{\frac{1}{2}}$, $\sqrt[3]{a}$ is written as $a^{\frac{1}{3}}$.

Also $\frac{1}{\sqrt{a}}=a^{-\frac{1}{2}}$, and $\frac{1}{\sqrt[3]{a}}=a^{-\frac{1}{3}}$.

Since $a^m \times a^n = a^{m+n}$ is true for all values of m and n. If n be 0, then

$$a^m \times a^0 = a^{m+0} = a^m;$$

$$\therefore \quad a^0 = \frac{a^m}{a^m} = 1.$$

$$(a^2)^3 = (a \times a)^3 = (a \times a)(a \times a)(a \times a) = a^6,$$

or in words, the cube of a^2 is equal to a^6.

The general form is $(a^n)^m = a^{mn}$.

Hence $$a^{\frac{m}{n}} = \sqrt[n]{a^m},$$

which is read as the n^{th} root of a to the power m.

Ex. 1. **Explain why the product is a^{12} when a^5 is multiplied by a^7, and why the quotient is a^5 when a^8 is divided by a^3.**

a^5 is a short way of writing $a \times a \times a \times a \times a$, and a similar meaning is attached to a^7;

$$\therefore \quad a^5 \times a^7 = a^{7+5} = a^{12}.$$

Also $$\frac{a^8}{a^3} = \frac{a \times a \times a \dots \text{ to 8 factors}}{a \times a \times a};$$

$$\therefore \quad \frac{a^8}{a^3} = a^{8-3} = a^5.$$

Ex. 2. **Simplify $\left(\frac{bc}{a}\right)^{m+n}\left(\frac{a}{c}\right)^m\left(\frac{a}{b}\right)^n$.**

Also find its value when $a=7$, $b=3$, $c=2$, $m=2$, $n=1$.

$$\left(\frac{bc}{a}\right)^{m+n}\left(\frac{a}{c}\right)^m\left(\frac{a}{b}\right)^n = \left(\frac{bc}{a}\right)^m\left(\frac{bc}{a}\right)^n\left(\frac{a}{c}\right)^m\left(\frac{a}{b}\right)^n$$

$$= \frac{a^m b^m c^m a^n b^n c^n}{a^m c^m a^n b^n} = b^m c^n.$$

Numerical value $= 3^2 \times 2 = 18$.

Ex. 3. **(i) Explain why $(a^2)^3 = (a^3)^2 = a^6$.**

(ii) If $2^{3x} \times 3^{2x} = 5184$, what is the value of x?

(ii) Simplify $(a^2b)^2 \times (ab^2)^3 \div (a^2b^2)^4$.

(i) $$(a^2)^3 = a \times a \times a \times a \times a \times a = (a^3)^2$$
$$= a^3 \times a^3 = a^{3+3} = a^6.$$

(ii) $$2^{3x} \times 3^{2x} = 5184;$$
$$\therefore \quad 8^x \times 9^x = 72^2,$$
$$\text{or} \quad 72^x = 72^2; \quad \therefore \quad x = 2.$$

(iii) $$(a^2b)^2 \times (ab^2)^3 \div (a^2b^2)^4 = \frac{a^4b^2a^3b^6}{a^8b^8} = \frac{1}{a}.$$

EXERCISES. XXV.

1. In what cases is $a^n - b^n$ divisible by $a+6$ and $a-b$? Write down the qnotient of $16x^4 - 81y^4$ by $2x+3y$.

Simplify the following expressions:

2. $a^{2p+q} \times a^{p+4q} \div a^{q-p}$.

3. $\dfrac{a^{m-n} \times a^{m-3n} \times a^{m-5n}}{a^{n-m} \times a^{n-3m} \times a^{n-5m}}$.

4. $\dfrac{27a^{m+p} \times b^{n-q}}{81a^m b^n}$

5. $\sqrt[13]{\{x^3y^2\sqrt[3]{[xyz^4\sqrt{(x^{-1}y^{-2}z^{-3})}]}\}^{12}}$.

6. $\left\{\dfrac{\sqrt[3]{a}}{\sqrt[4]{b^{-1}}}\left(\dfrac{b^{\frac{1}{4}}}{a^{\frac{1}{3}}}\right)^2 \div \dfrac{a^{-\frac{1}{3}}}{b^{-\frac{1}{2}}}\right\}^6$. Find its value when $b=4$

7. $a^{p+3q} \times a^{p+q} \div a^{q-p}$.

8. $\left(\dfrac{a^{\frac{3}{2}}b^{\frac{7}{4}}}{a^{\frac{2}{3}}b^{\frac{1}{2}}}\right)^{-\frac{1}{5}} \times \{\sqrt[3]{(a^{-2})}\sqrt[6]{(b^{-1})}\}^2$.

9. $\dfrac{a^{n-m} \times a^{n-2m} \times a^{n-3m}}{a^{m-n} \times a^{m-2n} \times a^{m-3n}}$.

MISCELLANEOUS EXERCISES. XXVI.

1. Divide (i) x^7+2187 by $x+3$;
   (ii) $1+3x$ by $1-2x$ to five terms in the quotient.

2. For what values of n is x^n+a^n exactly divisible by $x+a$?

3. Divide $1-5x^4+4x^5$ by $(1-x)^2$.

4. Divide $21x^5-2x^4-70x^3-23x^2+33x+27$ by $7x^2+4x-9$.

5. Divide $6x^4+x^2y+2x^2-12y^2+37y-28$ by $2x^2+3y-4$, and verify your result by putting $x=2$, $y=-1$, in divisor, dividend, and quotient.

6. Divide $x^3+8y^3-27z^3+18xyz$ by $x+2y-3z$, and test your answer by substituting $x=5$, $y=-4$, $z=3$, in the dividend, divisor, and quotient.

7. Multiply $x^2-ax+2a^2$ by $x^2+3ax+4a^2$, and divide
   $8x^4-2ax^3+3a^2x^2-2a^3x+a^4$ by $2x^2+ax+a^2$.

8. Multiply $3x^4-x^2-1$ by $2x^4-3x^2+7$, and divide
   $12x^4-17x^3-9x^2+13x-63$ by $4x^2-3x+7$.

9. Resolve into factors:
   (i) $x^2-14x-72$. (ii) $x^2+3ax-x-3a$. (iii) $3x^3-27x$.

10. Simplify

$$\text{(i)}\ \frac{x^2+x-6}{x^2+2x-8} \times \frac{x^2+4x}{x^2-9}.$$

$$\text{(ii)}\ \frac{4}{x^2+x} + \frac{2x-5}{x^2-x+1} - \frac{2x^2-11}{x^3+1}.$$

11. Multiply $x^2-4xy+6y^2$ by $x-5y$, and divide $5x^6-6x^5+1$ by x^2-2x+1.

12. Resolve into factors :

(i) $10x^2+x-2$.

(ii) $x^2-2x-255$.

(iii) $6x^2-9ax+4bx-6ab$.

13. Simplify

(i) $\frac{a}{a+x}+\frac{1}{2}\cdot\frac{a+x}{a-x}-\frac{1}{2}\cdot\frac{a^2+x^2}{a^2-x^2}$.

(ii) $\left(x-\frac{xy-y^2}{x+y}\right)\left(x-\frac{xy^2-y^3}{x^2+y^2}\right)\div\left(1-\frac{xy-y^2}{x^2}\right)$.

14. Simplify the expression

$$\frac{ax+1+\dfrac{az}{yz+1}}{x-\dfrac{z}{yz+1}}\div a+\frac{1}{x+\dfrac{1}{y+\dfrac{1}{z}}}.$$

15. Define a factor, a term, a power, a root as used in Algebra. From a rod a ft. long $b-c$ ft. are cut off; express in two ways, with and without brackets, the number of feet that are left.

16. Find the numerical values of the following expressions when $x=-1$, $y=-2$, $z=\frac{1}{2}$:

(i) $2x-\{9y-8x+2z-(4x+y)\}$.

(ii) $(x+y-z)^2+(x+y)^2(x-y+z)+(x-y)^3$.

17. (i) Simplify $(xy-1)(x^2y^2+xy+1)+(xy+1)(x^2y^2-xy+1)$.

(ii) Divide $\frac{1}{8}x^3+\frac{1}{27}y^3$ by $\frac{1}{2}x+\frac{1}{3}y$.

18. (i) Simplify $\frac{x+1}{x^2-x}-\frac{x+2}{x^2-1}-\frac{1}{x^3+1}$.

(ii) Substitute $\frac{a}{a-1}$ for x, and $\frac{2a}{2a+1}$ for y, in $\frac{xy}{x-y}$, and reduce the result to its simplest form.

19. (i) Simplify $\left(\frac{x}{x+1}-\frac{x-1}{x}\right)\div\left(\frac{x}{x+1}+\frac{x-1}{x}\right)$.

(ii) Substitute

$\frac{a+1}{ab+1}$ for x, and $\frac{ab+a}{ab+1}$ for y, in $\frac{x+y-1}{x-y+1}$,

and reduce the result to its simplest form.

(iii) If $3x=2a$, find the value of $(3a-2x)^2+\frac{a^2}{9}-2x^2$ in terms of a.

Summary.

Addition is performed by adding together like terms.

Subtraction.—Alter the signs of the quantities to be subtracted, and proceed as in addition.

Multiplication.—The product of two terms with like signs is positive, with unlike signs the product is negative. To multiply two simple expressions together, multiply the co-efficients and add the indices of like terms.

Division.—The object of division is to find how often one quantity (the divisor) is contained in another (the dividend), or the product of the divisor and the quotient is equal to the dividend.

Factors.—When an algebraic expression consists of the product of two or more quantities, each of those quantities is called a *factor* of the expression.

Index.—The number placed near the top and to the right of a quantity, and which expresses the power of a quantity, is called the *index.*

Multiplication.—To multiply together different powers of the same quantity, it is only necessary to add the index of one to the index of the other.

Division of different powers of the same quantity is performed by subtracting the index of one from the index of the other.

$a^n + b^n$ is divisible by $a + b$ when n is an odd number.

$a^n - b^n$,, ,, $a - b$,, is odd.

$a^n - b^n$,, ,, both $a + b$ and $a - b$ when n is even.

The product of the sum and difference of two numbers is equal to the difference of their squares.

CHAPTER VII.

MEASURES AND MULTIPLES. H.C.F. AND L.C.M. FRACTIONS.

WE have already found that

$$(a+b)^2=a^2+2ab+b^2,$$

and by multiplying again by $a+b$ we obtain

$$(a+b)^3=a^3+3a^2b+3ab^2+b^3.$$

It is seen at once that some definite arrangement of the coefficients and indices can be made so that another power, say $(a+b)^4$, can be written down: the method used, and called the *Binomial Theorem*, is very important. It should be applied to the operation of expanding several simple expressions, such as $(a+b)^3$, $(a+b)^4$, etc., and afterwards committed to memory.

$$(a+b)^n=a^n+\frac{na^{n-1}b}{1}+\frac{n(n-1)}{1.2}a^{n-2}b^2+\dots .$$

Take $n=2$, then $a^n=a^2$, and $\dfrac{na^{n-1}b}{1}=2ab$.

Hence $(a+b)^2=a^2+\dfrac{2ab}{1}+\dfrac{b^2.(2.1)}{1.2}=a^2+2ab+b^2.$

Take $n=3$; here $a^n=a^3$; $\dfrac{na^{n-1}b}{1}=3a^2b$, etc.;

$$\therefore\ (a+b)^3=a^3+\frac{3a^2b}{1}+\frac{3.2ab^2}{1.2}+\frac{3.2.1}{1.2.3}a^0b^3$$

$$=a^3+3a^2b+3ab^2+b^3.$$

Writing down terms in the numerator which are afterwards cancelled by corresponding numbers in the denominator, would appear to the beginner to be a tedious and unnecessary process, but to avoid mistakes it is better to write out in full, as above, and afterwards to cancel common factors in numerator and denominator.

The expansion of

$$(1+a)^n \text{ is } 1+\frac{na}{1}+\frac{n(n-1)a^2}{1.2}+\ldots$$

if a is a very small quantity, the two first terms are sufficient for all practical purposes; thus, when a is small,

$$\mathbf{(1+a)^n=1+na} \text{ approximately.}$$

If $a=\cdot 01$ and $n=2$, then $(1+\cdot 01)^2=1+2\times\cdot 01=1\cdot 02$.

Ex. 1. $(1+\cdot 05)^2=1+2\times\cdot 05=1\cdot 1$; more accurately $(1\cdot 05)^2=1\cdot 1025$.

Ex. 2. $(1+\cdot 05)^3=1+3\times\cdot 05=1\cdot 15$.

Ex. 3. $\sqrt[3]{(1\cdot 05)}=(1+\cdot 05)^{\frac{1}{3}}=1+\frac{1}{3}\times\cdot 05=1\cdot 0167$.

Ex. 4. $\frac{1}{\sqrt[3]{1\cdot 05}}=(1+\cdot 05)^{-\frac{1}{3}}=1-\frac{1}{3}\times\cdot 05=1-\cdot 0167=\cdot 9833$.

EXERCISES. XXVII.

1. When a is very small we may use the relation $(1+a)^n=1+na$; by means of this rule find the values of $\sqrt[3]{1\cdot 003}$ and $(\cdot 996)^2$, and find the error per cent. in the latter case.

Find by the approximate rule the value of the following :-

2. $\sqrt[3]{\cdot 99}$.
3. $(1\cdot 001)^3$.
4. $(1\cdot 01)^{-\frac{1}{3}}$.
5. $(1\cdot 004)^2$.
6. $(1\cdot 004)^{-\frac{1}{2}}$.
7. $(1\cdot 01)^{-\frac{1}{4}}$.
8. $(1\cdot 01)^4$.
9. $\sqrt{(1\cdot 08)}$ and $\sqrt[4]{(1\cdot 08)}$.
10. $\sqrt{\cdot 92}$ and $\sqrt[4]{\cdot 92}$.
11. $\frac{1}{\sqrt{(1\cdot 08)}}$ and $\frac{1}{\sqrt{\cdot 92}}$.
12. $\frac{1}{\sqrt[4]{1\cdot 08}}$; $\frac{1}{\sqrt[4]{\cdot 92}}$.
13. $\sqrt[3]{(1\cdot 09)}$; $\sqrt[3]{\cdot 91}$; $\frac{1}{\sqrt[3]{1\cdot 09}}$; $\frac{1}{\sqrt[3]{\cdot 91}}$.

Measure and Multiple.—When one quantity is exactly divisible by another, the former is said to be a multiple of the latter. Thus $6bc$ is a multiple of b, $6c$, $6b$, c, and bc, because all these are contained in $6bc$ without a remainder.

It is necessary to distinguish between *multiple* and *least common multiple.* Thus the product of $4c$, $6a$, $9a^2bc$ and $12ab^2c$ is $2592a^4b^3c^3$, and obviously each of the given quantities can be divided into this product without a remainder; but it is evident that this is not the lowest such quantity, as each quantity is divisible into $36a^2b^2c$, and on trial it will be found that this is also the lowest quantity.

Common Measure.—Any number which exactly divides two or more given numbers is called a common measure, thus 6 is a common measure of 12, 24, and 36.

Highest Common Factor.—*The highest common factor of two or more algebraical expressions is the expression of highest dimensions which can be divided into each of the given expressions without a remainder.*

The letters H.C.F. are usually used to denote the highest common factor. The term *greatest common measure* or G.C.M. is sometimes used (p. 41), but the H.C.F. is much more suitable when applied to algebraical expressions.

The H.C.F. of any given algebraical expression can be obtained by methods similar to those adopted to find the G.C.M. in Arithmetic. Thus, for example, to reduce $\frac{x^2+2x-3}{x^2+5x+6}$ to its simplest terms we may proceed in either of two ways:

(1) Obtain the factors of numerator and denominator and then divide both numerator and denominator by any common factor. When such factor is the highest common factor the fraction is said to be in its lowest terms or its simplest form.

(2) Find the H.C.F. of the numerator and denominator by the usual process of dividing one expression by the other, and divide both expressions by it.

Ex. 1. Reduce to its lowest terms the fraction

$$\frac{x^4-13x^2+36}{x^4-x^3-7x^2+x+6}.$$

The factors of x^4-13x^2+36 are (x^2+x-6) and (x^2-x-6), and of $x^4-x^3-7x^2+x+6$ are $(x^2-1)(x^2-x-6)$.

Therefore the H.C.F. is x^2-x-6.

$$\frac{x^4-13x^2+36}{x^4-x^3-7x^2+x+6}=\frac{(x^2+x-6)(x^2-x-6)}{(x^2-1)(x^2-x-6)}.$$

Dividing numerator and denominator by the H.C.F., we obtain

$$\frac{x^4-13x^2+36}{x^4-x^3-7x^2+x+6}=\frac{x^2+x-6}{x^2-1}.$$

If the given expressions are not readily factorized, the H.C.F. can be obtained by a method similar to that used in Arithmetic, as described on p. 42. The working is as follows:

$$\begin{array}{l}
x^4-13x^2+36\,)\,x^4-x^3-7x^2+x+6\,(\,1 \\
\qquad\qquad\qquad\; \underline{x^4 \qquad -13x^2 \quad +36} \\
\qquad\qquad\qquad\; -x^3+6x^2+x-30\,)\,x^4 \qquad -13x^2 \qquad +36\,(-x-6 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\; \underline{x^4-6x^3-x^2+30x} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad 6x^3-12x^2-30x+36 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \underline{6x^3-36x^2-6x+180} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad 24x^2-24x-144 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\; =24(x^2-x-6)
\end{array}$$

$$\begin{array}{l}
x^2-x-6\,)\,-x^3+6x^2+x-30\,(\,-x+5 \\
\qquad\qquad\;\; \underline{-x^3+x^2+6x} \\
\qquad\qquad\qquad\quad 5x^2-5x-30 \\
\qquad\qquad\qquad\quad \underline{5x^2-5x-30}
\end{array}$$

x^2-x-6 is H.C.F.

Ex. 2. Find the H.C.F. of x^3+3x^2-4 and x^4+x^3+2x-4.

$$\begin{array}{l}
x^3+3x^2-4\,)\,x^4+x^3+2x-4\,(\,x-2 \\
\qquad\qquad\quad\;\; \underline{x^4+3x^3-4x} \\
\qquad\qquad\qquad -2x^3 \qquad +6x-4 \\
\qquad\qquad\qquad \underline{-2x^3-6x^2 \qquad +8} \\
\qquad\qquad\qquad\qquad 6x^2+6x-12 \\
\qquad =6(x^2+x-2);
\end{array}$$

$$\begin{array}{l}
x^2+x-2\,)\,x^3+3x^2-4\,(\,x+2 \\
\qquad\qquad\quad \underline{x^3+x^2-2x} \\
\qquad\qquad\qquad 2x^2+2x-4 \\
\qquad\qquad\qquad \underline{2x^2+2x-4}
\end{array}$$

Therefore x^2+x-2 is the H.C.F. required.

Ex. 3. Find the H.C.F. of

$$7x^3-10x^2-7x+10 \text{ and } 2x^3-x^2-2x+1.$$

Here $$\begin{aligned} 7x^3-10x^2-7x+10 &= 7x(x^2-1)-10(x^2-1) \\ &= (7x-10)(x^2-1) \end{aligned}$$

Also $$\begin{aligned} 2x^3-x^2-2x+1 &= x^2(2x-1)-1(2x-1) \\ &= (x^2-1)(2x-1). \end{aligned}$$

Hence the H.C.F. is x^2-1.

Ex. 4. Simplify $\dfrac{7x^3-10x^2-7x+10}{2x^3-x^2-2x+1}$.

The H.C.F. is x^2-1.

$$\therefore\ \frac{7x^3-10x^2-7x+10}{2x^3-x^2-2x+1}=\frac{(x^2-1)(7x-10)}{(x^2-1)(2x-1)}=\frac{7x-10}{2x-1}.$$

Ex. 5. Reduce to lowest terms

$$\frac{x^4+4x^3-19x^2-46x+120}{x^2-25x^2+144}.$$

In the usual manner, we find the H.C.F. to be x^2+x-12. Hence we may write the fraction in the form

$$\frac{(x^2+3x-10)(x^2+x-12)}{(x^2-x-12)(x^2+x-12)}=\frac{x^2+3x-10}{x^2-x-12}=\frac{(x-2)(x+5)}{(x+3)(x-4)}.$$

Ex. 6. Prove $\dfrac{15x^2+xy-2y^2}{9x^2+3xy-2y^2}=\dfrac{5x+2y}{3x+2y}$;

$$15x^2+\ xy-2y^2=(5x+2y)(3x-y);$$

$$9x^2+3xy-2y^2=(3x+2y)(3x-y).$$

Hence $$\frac{(5x+2y)(3x-y)}{(3x+2y)(3x-y)}=\frac{5x+2y}{3x+2y}.$$

EXERCISES. XXVIII.

Find the highest common factor, or greatest common measure of

1. $x^4-2x^3+x^2-1$ and x^4-3x^2+1.
2. $x^2+11x+30$ and $x^3+8x^2+9x-18$.
3. x^8-1 and $x^{19}-1$.
4. $8x^3+6x^2-12x-9$ and $24x^4-20x^2-24$.
5. $6x^4-2x^3-17x^2+2x-5$ and $6x^5-4x^4-11x^3-3x^2-3x-1$.
6. $3x^3+10x^2+7x-2$ and $3x^3+13x^2+17x+6$.
7. $6x^3+x^2-11x-6$ and $6x^4-x^3-14x^2-x+6$.
8. $3x^4-3x^3-6x^2$ and $6x^4-12x^3+6x-12$.
9. $x^4-21x+8$ and $8x^4-21x^3+1$.
10. x^4+3x^2+x+3 and $2x^3-16x+6$.
11. x^4-1 and $x^4+2x^3+4x^2+2x+3$.
12. $x^2-6x-27$ and $2x^2-11x-63$.
13. x^3-3x^2+4x-2 and x^3+2x^2-4x+1.
14. $6x^3-17x^2+22x-15$ and $8x^3-22x^2+27x-18$.
15. $x^3-3x^2-13x+15$ and $x^3+9x^2+11x-21$.
16. $7x^4-10x^3+3x^2-4x+4$ and $8x^4-13x^3+5x^2-3x$.

17. $x^3y+2x^2y^2+2xy^3+y^4$ and $5x^5+10x^4y+5x^3y^2$.

18. $x^4+7x^3+6x^2-32x-32$ and $x^2+9x+20$.

19. $6x^3-10x^2+10x-4$ and $9x^3+3x^2+3x-6$.

20. $6a^3-5a^2x-6ax^2$; $4a^3x-12a^2x^2+9ax^3$ and $6a^2x-13ax^2+6x^3$.

Lowest Common Multiple.—The lowest common multiple of two or more algebraical expressions is *the expression of lowest dimensions into which each of the given expressions can be divided without a remainder.* The three letters L.C.M. are used to denote the lowest common multiple.

In order to contain each of two or more given expressions the L.C.M. must be the product of

(1) The L.C.M. of the coefficients;

(2) The lowest power of each factor which is divisible by every power of that factor which occurs in the given expressions.

Or, to find the L.C.M. of two given expressions, divide the product of the quantities by their H.C.F. Or better, find their H.C.F., divide one quantity by it, and multiply the quotient by the other.

Ex. 1. Find L.C.M. of $15x^3-15$, and $6x^2+12x-18$.

Resolving into factors

$$15x^3-15=15(x^3-1)=3\times 5(x-1)(x^2+x+1);$$

$$6x^2+12x-18=6(x^2+2x-3)=2\times 3(x+3)(x-1);$$

$$\text{H.C.F.}=2\times 3\times 5(x-1).$$

$$\text{L.C.M.}=3\times 5\times 2(x^2+x+1)(x+3)(x-1).$$

Ex 2. Find the L.C.M. of $7x^3-3x^2-7x+3$ and $2x^3-5x^2-2x+5$.

$$7x^3-3x^2-7x+3=x^2(7x-3)-1(7x-3)=(x^2-1)(7x-3);$$

$$2x^3-5x^2-2x+5=x^2(2x-5)-1(2x-5)=(x^2-1)(2x-5);$$

$$\therefore\ \text{L.C.M. is } (x^2-1)(2x-5)(7x-3).$$

Ex. 3. Find the L.C.M. of $2x^2-5x+3$, $3x^2-x-2$, $6a^2-5x-6$.

$$2x^2-5x+3=(2x-3)(x-1).$$

$$3x^2-x-2=(3x+2)(x-1).$$

$$6x^2-5x-6=(2x-3)(3x+2).$$

$$\therefore\ \text{L.C.M.}=(3x+2)(2x-3)(x-1).$$

To find the L.C.M. of three or four or more quantities, find the L.C.M. of two of the quantities, then find the L.C.M. of this quantity and one of the remaining quantities, and so on.

Ex. 4. Find the L.C.M. of

$$4x^2+8x-12,\ 9x^2-9x-54,\ 6x^4-30x^2+24.$$

$$4x^2+8x-12=4(x^2+2x-3)=4(x+3)(x-1).$$

$$9x^2-9x-54=9(x^2-x-6)=9(x-3)(x+2).$$

$$6x^4-30x^2+24=6(x^4-5x^2+4)=6(x^2-4)(x^2-1).$$

$$\therefore\quad \text{L.C.M.}=36(x^2-4)(x^2-1)(x^2-9).$$

EXERCISES. XXIX.

Find the lowest common multiple (L.C.M.) of

1. $a^2-x^2,\ a^2+3ax+2x^2,\ a^2-4x^2.$
2. $a^4b^2,\ b^3c,\ a^2c^3$; also of
$$3x^2-10xy+3y^2,\ 3x^2-4xy+y^2,\ x^2-4xy+3y^2.$$
3. $x^3+2ax^2+a^2x+2a^3$ and $x^3-2ax^2+a^2x-2a^3$.
4. Show that the product of any two quantities is the same as the product of their L.C.M. and G.C.M.

Find the L.C.M. of

5. $32x^3y^3(x-y)^2,\ 18x^2y(x^3-y^3),\ 9xy^2(x^2+xy+y^2).$
6. $2x^4+9x^2+5x+12$ and $2x^4+4x^3+13x^2+11x+12$.
7. $x^2-5x+6,\ x^2-4x+3,\ x^2-3x+2.$
8. $15a^3x(a+x)^3,\ 20ax^3(a-x)^3,\ 36a^2x^2(a^2-x^2)^2.$
9. $x^2-y^2,\ x^3-y^3$, and x^3+y^3.
10. $3a^4xy,\ 5abx^3y^3$, and $6b^3xy^4$.
11. $24a^6(a^3-x^3),\ 25a^5x^2(a^2-x^2)$, and $60a^4x^5$.
12. $12x^4(x^2+2xy+y^2),\ 15x^3y(x^2-2xy+y^2)$, and $16x^2y^2(x^2-y^2)$.
13. $10x^2y^2(x^3-y^3),\ 15y^4(x-y)^3$, and $12x^3y(x-y)(x^2-y^2)$.
14. $243(x^4y-x^3y^2),\ 324(x^4y-x^2y^3)$, and $432(x^4y-xy^4)$.
15. $27(x^2y^7-5xy^8+6y^9),\ 33(x^3y^3-4x^2y^4+3xy^5)$, and
$$36(x^3y^6-3x^2y^7+2xy^8).$$

Fractions.—The rules and methods adopted in dealing with fractions in Algebra are almost identical with those already considered in Arithmetic. In both cases fractions are of frequent occurrence and their consideration is of the utmost importance. Some little practice is necessary before even a simple fraction can be reduced to its lowest terms. Perhaps the best method in the simplification of fractions is, as already shown, to write out wherever possible the given expressions in factors. To do this readily and easily, the factors already referred to on page 88 should be carefully learnt by heart. If this simplification cannot be done by inspection, the H.C.F. of the numerator and denominator should be found, and from this, as shown on p. 101, the L.C.M. is readily obtained.

When required to add, subtract, or compare fractional expressions, it is necessary that they shall all have a common denominator, and to lessen the work it is desirable that this denominator shall be as small as possible.

Ex. 1. Simplify $\left(\frac{1}{2}+\frac{1}{3x}\right)\div\left(9x-\frac{4}{x}\right)$.

$$\frac{\frac{1}{2}+\frac{1}{3x}}{9x-\frac{4}{x}}=\frac{\frac{3x+2}{6x}}{\frac{9x^2-4}{x}}=\frac{x(3x+2)}{6x(3x+2)(3x-2)}=\frac{1}{6(3x-2)}.$$

Ex. 2. Simplify $\frac{x-a}{\frac{1}{a}-\frac{1}{b}}\times\frac{a-b}{1-\frac{a}{x}}$.

$$\frac{x-a}{\frac{1}{a}-\frac{1}{b}}\times\frac{a-b}{1-\frac{a}{x}}=\frac{x-a}{\frac{b-a}{ab}}\times\frac{a-b}{\frac{x-a}{x}}$$

$$=\frac{(x-a)ab}{b-a}\times\frac{(a-b)x}{x-a}=-abx.$$

Ex. 3. Write as a fraction in its lowest terms the value of

$$\frac{1}{x^3-3x^2-15x+25}-\frac{1}{x^3+7x^2+5x-25}.$$

In this example either find the L.C.M. or write the expressions in factors, thus:

$$\frac{1}{(x^2+2x-5)(x-5)}-\frac{1}{(x+5)(x^2+2x-5)}.$$

Hence the L.C.M. is $(x-5)(x+5)(x^2+2x-5)$;

$$\therefore\quad\frac{1}{(x^2+2x-5)(x-5)}-\frac{1}{(x+5)(x^2+2x-5)}$$

$$=\frac{x+5-(x-5)}{(x-5)(x+5)(x^2+2x-5)}=\frac{10}{(x-5)(x+5)(x^2+2x-5)}.$$

Ex. 4. Simplify $\frac{x-2y}{x+2y}+\frac{x+2y}{x-2y}+\frac{8xy}{x^2-4y^2}$.

Here the L.C.M. is x^2-4y^2;

$$\therefore\ \text{the given fraction}=\frac{(x-2y)^2+(x+2y)^2+8xy}{x^2-4y^2}$$

$$=\frac{2x^2+8xy+8y^2}{x^2-4y^2}=\frac{2(x+2y)(x+2y)}{(x+2y)(x-2y)}=\frac{2(x+2y)}{x-2y}.$$

EXERCISES. XXX.

Simplify the following expressions :

1. $\frac{x+1}{x^2-2x}\times\frac{x+2}{x^2-x}\div\frac{x^2-4}{x^4-x^2}$.

2. $\frac{x}{9}+\frac{2}{3}+\frac{4}{x-6}-\frac{2x}{3(x-6)}$.

3. $\frac{9a^3yz}{8bcx^3}\times\frac{4b^3zx}{15cay^3}\times\frac{10c^3xy}{abz^3}$.

4. $\frac{x^4-a^4}{x^3-a^3}\times\frac{x^2+ax+a^2}{x+a}$.

5. $\frac{3x-5}{2}+\frac{2x-4}{3}+3\frac{5}{6}-\frac{x}{6}$.

6. $\left(1-\frac{2a}{1+a}\right)\times\left(1+\frac{2a}{1-a}\right)$.

7. $\frac{a-x}{ax}-\frac{a+x}{x^2}+\frac{x}{a^2}+\frac{4a^2+x^2}{4ax^2}$.

8. $\frac{a^2-ax+x^2}{6a}\times\frac{4a^2x^2}{a-x}\div\frac{a^3+x^3}{a^2-x^2}$.

9. $\frac{3x^3+15ax^2+28a^2x-4a^3}{12a^2x}-\frac{x^2+5ax+4a^2}{4a^2}+\frac{a-x}{3x}$.

10. $\frac{32(a^3-x^3)}{a^2+x^2}\times\frac{a+x}{a-x}\div\frac{8(a^2+ax+x^2)}{a^2+x^2}$.

11. $\frac{3(5+2x)}{xy}-\frac{3ax+by+15a}{axy}+\frac{b}{ax}$.

12. $\frac{25x(x^2-y^2)}{36(x-y)^2}\times\frac{12x(x-y)}{x^2+xy}\div\frac{5}{3x^2}$.

13. $\frac{a^3+abx-bxy}{a(a^2-y^2)}-\frac{bx}{a(a+y)}-\frac{a}{2(a-y)}$.

14. $\left(2x-\frac{x^2-y^2}{x}\right)\left(3y+\frac{x^2+y^2}{y}\right)\div\left(\frac{x^2}{y^2}+5+\frac{4y^2}{x^2}\right)$.

15. $a^2(b-c)+b^2(c+a)+c^2(a-b)+(b-c)(c-a)(a-b)$.

16. $\frac{(ab-a-b+1)(ab+a+b+1)}{(ab-2a+b-2)(ab-a-2b+2)}$.

17. $\frac{2}{1-x}-\frac{3x-1}{2+2x}+\frac{2-x^2}{1-x^2}$, and find its value when $3x^2=1$.

18. Substitute $\frac{a-1}{a+1}$ for x and $\frac{2a-1}{2a+1}$ for y in the expression $\frac{x-y}{x+y}$, and reduce the result to its simplest form.

19. $\left[\frac{1+x}{1-x}-\frac{1-y}{1+y}\right]\div\left[1+\frac{(1+x)(1-y)}{(1-x)(1+y)}\right]$.

20. $\left(1-\frac{1}{1+x}-\frac{x}{1-x^2}+\frac{x^2}{1+x^3}\right)\times\frac{x-1}{x}$.

21. $\frac{1}{x-\frac{x}{x+1}}+\frac{1}{x+\frac{x}{x-1}}+\frac{1-\frac{3}{x}}{1-\frac{x}{3}}$.

22. $\frac{a+b}{a-b}-\frac{2ab}{a^2-b^2}-\frac{2a^2b^2}{a^4-b^4}$.

23. $x+y-\frac{9x^2-4y^2}{3x+2y}$.

24. $\left(\frac{1}{2a+b}+\frac{1}{2a-b}-\frac{3a}{4a^2-b^2}\right)\times\frac{4a^2+4ab+b^2}{2a-b}$.

25. $\dfrac{\frac{x}{x+1}+\frac{x}{x-1}}{\frac{2}{x^2-1}}-\dfrac{4x-\frac{1}{x}}{2+\frac{1}{x}}$.

26. $\left(\frac{1}{x^3-y^3}-\frac{1}{x^3+y^3}\right)\div\left(\frac{1}{x-y}-\frac{1}{x+y}-\frac{2y}{x^2+y^2}\right)$.

27. $\dfrac{2x}{1-\frac{x^2}{y^2}}+\dfrac{y}{1-\frac{y}{x}}$.

28. $\left(x^2-\frac{2x}{x-1}\right)\times\frac{x^3-1}{x^2-1}\div(x^3+x^2+x)$.

Evolution.—The root of any given expression is that quantity which, when multiplied by itself the required number of times, produces the given expression. The operation by which the roots of quantities are obtained is called *evolution.*

Square Root.—In many simple cases the root required is evident on inspection.

Ex. 1. $\sqrt{64a^6x^6}$, this may be written $(8a^3x^3)^2$, and the required root is $8a^3x^3$.

But both $-8a^3x^3$ and $+8a^3x^3$ would, when squared, give $64a^6x^6$. Every positive quantity has two square roots, equal in magnitude but opposite in sign, and the above facts would be written $\sqrt{64a^6x^6}=\pm 8a^3x^3$.

Again, $\sqrt{9}=\pm 3$, the two signs indicating that either $+3$ or -3 is the root of 9.

Also, $\sqrt{a^2+2ab+b^2}=\pm(a+b)$.

When an expression can be arranged in three terms of the form $a^2+2ab+b^2$ (in which the middle term is twice the product of the square roots of the other two), the square root can be written down at once.

Ex. 2. $x^2+4x+4=(x+2)^2$.

Ex. 3. $x^2+8x+16=(x+4)^2$.

In expressions where the square root cannot be determined by inspection, we require to employ a rule which will probably be understood most readily by solving a simple example.

Ex. 4. Find the square root of x^2+6x+9.

Here x is clearly the root of the first term x^2.

Arrange the terms according to the powers of one term x. Put x for the first term of the root, square it, and subtract the square from the given expression. Bring down the other two terms $6x+9$. These can be written $3(2x+3)$. Thus the second term, 3, of the root will be obtained when the remainder $6x+9$ is divided by $2x+3$, or twice the first term added to the second.

$$\begin{array}{r|l}
 & x^2+6x+9\ (\ x+3 \\
 & x^2 \\
\hline
3(2x+3) & 6x+9 \\
 & \underline{6x+9}
\end{array}$$

Multiplying $2x+3$ by the second term 3, the result is $6x+9$; this subtracted from $6x+9$ leaves no remainder, hence the required root is $x+3$.

Following the steps in the above example, the following case will be readily understood:

Ex. 5. Extract the square root of $4a^4+9(1-2a)+3a^2(7-4a)$;

i.e. $4a^4+9-18a+21a^2-12a^3$.

$$\begin{array}{r|l}
 & 4a^4-12a^3+21a^2-18a+9\ (\ 2a^2-3a+3 \\
 & 4a^4 \\
\hline
4a^2-3a\ ) & \quad -12a^3+21a^2 \\
 & \quad -12a^3+\ 9a^2 \\
\hline
4a^2-6a+3\ ) & \qquad\qquad 12a^2-18a+9 \\
 & \qquad\qquad \underline{12a^2-18a+9}
\end{array}$$

The method adopted is as follows:

(*a*) Clear away the brackets.

(*b*) Arrange the terms according to the ascending or descending powers of some term (in the given example the terms are arranged in descending powers of a).

(*c*) Take the square root ($2a^2$) of the first term for the first term of the square root required.

(*d*) Subtract the square of $2a^2$, and bring down the next two terms ($-12a^3+21a^2$).

(*e*) Multiply the first term of the square root by 2, and use it as a trial divisor, so obtaining the second term $-3a$, $\therefore\ -3a\times 4a^2=-12a^3$. Multiply $4a^2-3a$ by $-3a$ and obtain $-12a^3+9a^2$. By subtraction we obtain remainder $12a^2$.

(*f*) Bring down the next two figures, and by doubling the last term of the square root we obtain, as a trial divisor, $4a^2-6a$. Dividing by this trial divisor we obtain the third term 3. Hence $2a^2-3a+3$ is the square root required.

EXERCISES. XXXI.

Find the square root of each of the following expressions:

1. $25x^8 - 60x^6 - 34x^4 + 84x^2 + 49.$ 2. $x^4 + 8x^3 - 26x^2 - 168x + 441.$

3. $a^4 - 2a^3 + 9a^2 - 8a + 16.$ 4. $16x^4 - 40x^3 + 89x^2 - 80x + 64.$

5. $25x^4 - 30x^3 + 49x^2 - 24x + 16.$

6. $4a^4 - 12a^3x + 13a^2x^2 - 6ax^3 + x^4.$

7. $4x^4 - 4x^3 + 3x^2 - x + \frac{1}{4}.$ 8. $x^6 + 6x^4 - 4x^3 + 9x^2 - 12x + 4.$

9. $\dfrac{y^2}{x^2} + 2\dfrac{y}{x} + 3 + 2\dfrac{x}{y} + \dfrac{x^2}{y^2}.$ 10. $\dfrac{4x^4}{9} - 8x^3 + \dfrac{112}{3}x^2 - 12x + 1.$

11. $x^4 + \dfrac{1}{x^4} + 2\left(x^2 + \dfrac{1}{x^2}\right) + 3.$ 12. $\dfrac{4x^2}{y^2} + \dfrac{4y^2}{x^2} - 8.$

13. $9x^4 - 30x^3y + 31x^2y^2 - 10xy^3 + y^4.$

MISCELLANEOUS EXERCISES. XXXII.

Find the highest common factor of

1. $3x^4 + 7x^3 + 2x^2 - 31x - 35$ and $15x^4 + 2x^3 - 34x^2 - 34x - 21.$

2. $2x^4 - 3x^3 - 4x^2 - 9x - 6$ and $4x^3 - 9x^2 - 12x - 27$, and write down these expressions in factors.

3. $4x^4 - 4x^3 + x^2 - 1$ and $2x^3 + 5x^2 - 2x + 3$, and write down the expressions in factors.

4. $6x^4 - 2x^3 + 7x^2 - x + 2$ and $6x^4 - 12x^3 + 21x^2 - 6x + 9.$

5. $x^4 - 5x^3 - 6x^2 + 35x - 7$ and $3x^3 - 23x^2 + 43x - 8$, and write down these expressions in factors.

6. $2x^4 + 9x^2 + 5x + 12$ and $2x^4 + 4x^3 + 13x^2 + 11x + 12.$

7. $45x^3y + 3x^2y^2 - 9xy^3 + 6y^4$ and $54x^2y - 24y^3.$

8. $3x^3 - 13x^2 + 23x - 21$ and $6x^3 + x^2 - 44x + 21.$

9. Find the greatest common measure of the numerator and denominator of the fraction

$$\frac{3x^2 - (4a + 2b)x + a^2 + 2ab}{x^3 - (2a + b)x^2 + (a^2 + 2ab)x - a^2b},$$

and reduce it to its lowest terms.

Simplify the following:

10. $\dfrac{1}{1 - \dfrac{x}{y}} + \dfrac{\dfrac{x}{y}}{1 - \dfrac{y}{x}}.$ 11. $\dfrac{x^8 + x^4y^4 + y^8}{x^6 - y^6}.$

Simplify the following:

12. $\left(\frac{x-y}{x+y}+\frac{x+y}{x-y}\right)\div\left(\frac{x}{y}+\frac{y}{x}\right).$ 13. $\frac{2}{y+3x}-\frac{1}{y-3z}+\frac{9x}{y^2-9x^2}.$

14. $\left(1-\frac{4xy}{(x+y)^2}\right)\div\frac{x^2-y^2}{(x+y)^3}.$ 15. $\frac{1}{a-2}+\frac{1}{a+2}-\frac{4}{a^2-4}$

16. $\frac{x^2-8x+7}{x^2-6x-7}.$ 17. $\frac{(m-a)(m+a)+n(2m+n)}{m+n+a}$

18. Simplify $2x^2-3\left(\frac{y^2}{2}-x^2\right)-\frac{1}{2}\left\{4x^2-\frac{1}{2}(24x^2+5y^2)\right\},$

and divide the simplified expression by $x+\frac{y}{6}.$

19. Reduce the following expressions to their simplest forms:

(i) $\frac{x+2y}{x-2y}-\frac{x-2y}{x+2y}.$ (ii) $\frac{1+\frac{a}{b}}{\frac{b}{a}-1}\div\frac{a^2-b^2}{1-\frac{2b}{a}+\frac{b^2}{a^2}}.$

Find the value of the former expression (i) when

$x=-1$ and $y=\frac{\bar{1}}{4}.$

20. If $\frac{x}{a}=\frac{y}{b}$, show that either fraction is equal to

$\frac{x+y}{a+b}$ and to $\frac{x-y}{a-b}.$

21. Reduce to its lowest terms the fraction

$$\frac{8x^3-10x^2-16x-3}{6x^4-22x^3+31x^2-23x-7},$$

and find its value when $x=-\frac{2}{3}.$

Find the factors of the following expressions, and write down the factors of their least common multiple:

22. (i) $2x^2+3x-2.$ (ii) $(5x-7)^2-(x-5)^2.$ (iii) $2x^3-x^2-8x+4.$

23. (i) $x^2-8x+12.$ (ii) $3x^2-20x+12.$ (iii) $3x^3-2x^2-12x+8.$

24. $4x^2+8x-12,$ $9x^2-9x-54,$ $6x^4-30x^2+24.$

25. $15x^3-15,$ $6x^2+12x-18,$ $10x^2-90.$

Write down the following expressions in factors:

26. (i) $15x^2-16xy-15y^2.$ (ii) $(a+b+c)^2-4(b-c)^2.$
(iii) $(1+x)^3-(1-x)^3.$

27. Find the value of $\frac{x+2a}{2b-x}+\frac{x-2a}{2b+x}+\frac{4ab}{x^2-4b^2}$, when $x=\frac{ab}{a+b}$;

and simplify the expression $\frac{b}{\sqrt{a}}\times\sqrt[3]{ac}\times\frac{\sqrt[4]{c^3}}{\sqrt{b}}\times\frac{\sqrt{b^{-1}}}{a^{-\frac{1}{4}}}.$

Summary.

Multiple.—When one quantity is exactly divisible by another, the former is said to be a multiple of the latter.

Common Measure.—A number which exactly divides two or more given numbers is called a measure, or common measure of the numbers.

Highest Common Factor.—The highest common factor (H.C.F.) of two or more algebraical expressions is the expression of highest dimensions which can be divided into each of the given expressions without a remainder; it is often called the *greatest common measure* (G.C.M.).

The Lowest Common Multiple (L.C.M.) of two or more algebraical expressions is the expression of lowest dimensions into which each of the given quantities can be divided without a remainder.

Evolution.—The root of a given quantity is (as in Arithmetic) defined as that quantity which, when multiplied by itself the required number of times, produces the given quantity.

CHAPTER VIII.

SIMPLE EQUATIONS. SIMULTANEOUS EQUATIONS.

Simple Equations.—When two algebraical expressions are connected together by the sign of equality the whole expression thus formed is called an *equation*, and the use of an equation consists in this, that from the relations expressed between certain known and unknown quantities we are able under proper conditions to find the unknown in terms of the known.

Usually the earlier letters of the alphabet, a, b, c, d..., are used to represent known, and the concluding letters, x, y, z, to represent unknown quantities.

The process of finding the value of the unknown quantity is called **solving the equation**, the value so found is the **solution** or the **root** of the equation.

An equation which involves the unknown quantity to the first power or degree is called a **simple equation**; if it contains the square of the unknown quantity it is called a **quadratic equation**; if the cube of the unknown quantity, a **cubic equation**.

If an equality involving only an algebraic operation exists between two quantities the expression is called an **identity**, thus $(x+y)^2=x^2+2xy+y^2$ is an identity.

In the example $4x+2=30$ the statement intended is that if 2 be added to four times an unknown number x, the result will equal 30. By a process of trial, substituting the numbers 1, 2, 3 in turn for x, it will be found that the equation is true only when $x=7$. Then as $4\times 7=28$, this value of x makes the expressions on the left- and right-hand sides of the sign of equality numerically equal, or, the equation is said to be satisfied. The solution of a simple equation depends upon the following truths.

If two quantities are equal to one another they remain equal when:

(*a*) The same quantity is added or subtracted from each side of the original equation.

(*b*) When each side is multiplied or divided by the same quantity.

(*c*) When each side is raised to the same power, or the same root of each is extracted.

(*d*) The signs of all the terms in the equated expressions can be changed from + to − if both sides are similarly altered.

Ex. 1. Solve $4x+2=30$.

By (*a*) subtracting 2 from each side we get $4x=28$;

$$\therefore\ x=\tfrac{28}{4}=7.$$

Instead of subtracting we can **transpose** the 2 in the preceding example from one side of the equation to the other by changing its sign; thus $4x=30-2=28$.

Ex. 2. Solve $4x+5=3x+8$.

Subtract $3x$ from both sides of the equation and we get

$$4x-3x+5=8,$$

next subtract 5 from each side;

$$\therefore\ x=8-5=3.$$

In the above it is obvious that $+3x$ and $+5$ on one side may be removed from one side to the other (or as it is called *transposed*) and appear on the opposite side with changed sign.

Hence in the solution of equations, *Transpose all the unknown quantities to one side, and all the known quantities to the other, simplify if necessary, and divide by the coefficient of the unknown quantity.*

The rules referred to above, under (*a*), (*b*), (*c*), (*d*), can perhaps be best illustrated by the consideration of a few simple examples.

Rule (*a*).

Ex. 3. Solve $4x+2=3x+4$;

transposing, $4x-3x=4-2$; $\therefore\ x=2$.

Ex. 4. Solve $5(x+1)=3(x-5)+2$,

or $5x+5=3x-15+2$;

transposing, $5x-3x=2-15-5$;

$$\therefore\ 2x=-18;\ \text{or}\ x=-9.$$

RULES (*b*) AND (*d*).

Ex. 5. Solve $\frac{x}{2}+\frac{x}{3}=x-7.$

Multiplying both sides by 6,

$$3x+2x=6x-42,$$

or

$$5x-6x=-42\,;$$

$$\therefore\quad -x=-42,\text{ or } x=42.$$

Ex. 6. Solve $\frac{x}{5}+\frac{16}{5}=\frac{2x}{3}-1.$

L.C.M. of denominators is 15. Hence multiplying both sides by 15,

$$3x+48=10x-15\,;$$

$$3x-10x=-15-48,$$

$$-7x=-63\,;\quad\therefore\quad x=9.$$

Ex. 7. Solve $\frac{a}{bx}-a^2=b^2-\frac{b}{ax}.$

First clear fractions by multiplying all through by *abx*;

$$\therefore\quad a^2-a^3bx=ab^3x-b^2,$$

transposing,

$$-a^3bx-ab^3x=-a^2-b^2,$$

changing sign or multiplying by -1,

$$x(a^2+b^2)ab=a^2+b^2,$$

$$\therefore\quad x=\frac{a^2+b^2}{ab(a^2+b^2)}=\frac{1}{ab}.$$

Abbreviated and Particular Methods.—In solving equations it often happens that an easy solution can be obtained by an advantageous arrangement of the terms on the two sides of an equation.

Ex. 8. Solve $\sqrt{x+4}+\sqrt{x-1}-5=0.$

Transposing, we get

$$\sqrt{x+4}=5-\sqrt{x-1}.$$

Squaring both sides,

$$x+4=25-2(5\sqrt{x-1})+x-1$$

$$=25-10\sqrt{x-1}+x-1.$$

Again transposing,

$$10\sqrt{x-1}=25-1-4=20\,;$$

$$\therefore\quad \sqrt{x-1}=2.$$

Again squaring, $x-1=4\,;\quad\therefore\quad x=5.$

It is well to notice that if we commence to solve the equation by squaring, the resulting equation will become very troublesome and difficult.

If in an equation each side consists of a simple fraction, the work in many cases is much simplified by adding or subtracting the numerator and denominator of each fraction to form in each case a new numerator, and subtracting or adding the numerator and denominator to form a new denominator.

Thus, if $$\frac{a}{b}=\frac{c}{d},$$

then by adding unity to each side of the equation, we get

$$\frac{a}{b}+1=\frac{c}{d}+1\,;$$

$$\therefore\quad \frac{a+b}{b}=\frac{c+d}{d}\,;$$

or, $$\frac{a+b}{c+d}=\frac{b}{d}\quad\ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}$$

In a similar manner, subtracting unity from each side,

$$\frac{a}{b}-1=\frac{c}{d}-1\,;$$

or, $$\frac{a-b}{b}=\frac{c-d}{d}\,;$$

$$\therefore\quad \frac{a-b}{c-d}=\frac{b}{d}\quad\ldots\ldots\ldots\ldots\ldots\ldots\text{(ii)}$$

Hence, from (i) and (ii), since things which are equal to the same thing are equal to one another,

$$\frac{a-b}{c-d}=\frac{a+b}{c+d}\,;$$

or, $$\frac{a+b}{a-b}=\frac{c+d}{c-d}\quad\ldots\ldots\ldots\ldots\ldots\ldots\text{(iii)}$$

Ex. 9. Solve $\dfrac{\sqrt{12+x}+\sqrt{12}}{\sqrt{12+x}-\sqrt{12}}=2$ or $\dfrac{2}{1}$.

By (iii) $$\frac{\sqrt{12+x}+\sqrt{12}+\sqrt{12+x}-\sqrt{12}}{\sqrt{12+x}+\sqrt{12}-(\sqrt{12+x}-\sqrt{12})}=\frac{2+1}{2-1}\,;$$

or $$\frac{2\sqrt{12+x}}{2\sqrt{12}}=3\,;$$

$$\therefore\quad \sqrt{12+x}=3\sqrt{12}.$$

Squaring, $$12+x=9\times 12\,;\quad \therefore\quad x=96,$$

Work out the following in the same way :

$$\frac{\sqrt{a}+\sqrt{a-x}}{\sqrt{a}-\sqrt{a-x}}=b. \quad \left(x=\frac{4ab}{(b+1)^2}\right).$$

Ex. 10. Solve $\frac{x-a}{x+a}+\frac{3b-x}{2b+x}=0.$

Multiplying all through by $(x+a)(2b+x)$, we get

$$(x-a)(2b+x)+(x+a)(3b-x)=0.$$

Multiplying out and collecting terms,

$$x(5b-2a)=-ab; \quad \therefore \quad x=\frac{-ab}{5b-2a}.$$

Ex. 11. Solve $\frac{3x-7}{5}-\frac{3x+7}{4}=\frac{5x-9}{8}-\frac{3x+9}{6}.$

Here the L.C.M. of the denominators is 120.

$$\therefore \quad 24(3x-7)-30(3x+7)=15(5x-9)-20(3x+9).$$

Multiplying out and collecting terms, we get

$$33x=-63; \quad \therefore \quad x=-\tfrac{21}{11}.$$

The solution can in many cases be easily effected if the equation can be factorized; the method of finding a value for x which satisfies has been already explained on p. 89.

Ex. 12. Solve $x^2-5x+6=0.$

$$x^2-5x+6=(x-2)(x-3).$$

Thus, either $x=2$ or $x=3$ satisfy the given equation, and these are the values required.

EXERCISES. XXXIII.

Solve the following equations :

1. $\frac{2x}{15}+\frac{x-6}{12}-\frac{3x}{20}=1\frac{1}{2}.$ 2. $\frac{1}{x+3}+\frac{1}{x-2}=\frac{2}{x-7}.$

3. $\frac{1}{2}(x-1)-\frac{1}{3}(2-x)+\frac{1}{4}(x+1)=x.$

4. $\frac{x}{2}-\frac{5x}{3}-\frac{4}{3}=\frac{4x}{3}-3.$ 5. $\frac{5}{2}-\frac{x+4}{11}=x+\frac{1}{2}.$

6. $\frac{x+3}{4}-\frac{x-3}{5}=\frac{x-5}{2}-2.$ 7. $\frac{2x-5}{4}-\frac{x-2}{6}=\frac{x}{7}-\frac{1}{4}.$

8. $\frac{3x+5}{7}-\frac{6x+5}{9}=x-\frac{2}{3}.$

9. $\frac{3x-13}{8}-\frac{4x+6}{9}=1-\frac{x-1}{10}.$

10. $\frac{2x-\frac{1}{3}}{5}-\frac{x+\frac{2}{3}}{3}+\frac{4}{15}=0.$

11. $\frac{a}{x-a}+\frac{x+a}{a}=\frac{x-b}{a}.$

12. $\frac{x}{6}-\frac{x-\frac{1}{2}}{3}-\frac{1}{3}\left(\frac{2}{5}-\frac{x}{3}\right)=0.$

13. $\frac{2x+a}{b}-\frac{x-b}{a}=\frac{3ax+(a-b)^2}{ab}.$

14. $x+\frac{9b^2}{a}=\frac{3bx}{a}+a.$

15. $\frac{x+a}{a}+\frac{x}{x-a}=\frac{x-a}{a}.$

16. $ax+b=3ax+c.$

17. $\frac{2}{x-a}-\frac{1}{x-b}=\frac{x}{x^2-a^2}.$

18. $5x-\{3x+6-(2x-11)\}=6x+11-\{2x-3-(15-2x)\}.$

19. $(ax+b)(bx-a)=a(bx^2-a).$

20. $\frac{x-1}{3}-\frac{2x-7}{4}=\frac{3}{4}.$

21. Find the relation between a and b when

$$\frac{3}{x-a}-\frac{2}{x+a}=\frac{x+b}{x^2-a^2}.$$

Solve:

22. $\frac{x-3}{2}+\frac{x-4}{3}=5.$

23. $\frac{x+m}{x-n}+\frac{x+n}{x-m}=2.$

24. $(2x-1)(3x-2)-(x-1)(x+1)=5(x-1)^2.$

Problems producing Simple Equations with one unknown quantity.—Having ascertained how to solve an equation involving one unknown quantity, it is only necessary to express the conditions of any given question by means of symbols, and thus to produce equations relating to number, quantity, shape, etc.

Ability to solve an equation readily and easily will be found of the greatest value in all, or nearly all, questions in applied science. A few questions and their solutions are given to show the method by which some required or unknown quantity, x, is obtained from an equation.

Ex. 1. The difference of two numbers is 14. If 9 times the less be subtracted from 6 times the greater, the remainder is 33. What are the numbers?

Let x denote the smaller number.

Then $x+14$ will represent the greater number.

Further, $9x$ is 9 times the smaller number, and $6(x+14)$ is 6 times the greater number.

But when 9 times the smaller is subtracted from 6 times the greater, the remainder is 33.

$$\therefore\ 6(x+14)-9x=33,$$

or $$6x+84-9x=33;$$

$$\therefore\ -3x=33-84=-51,$$

or $$x=\frac{-51}{-3}=17, \text{ the smaller number,}$$

and $$x+14=31, \text{ the greater number.}$$

Ex. 2. What number is that to which, if 8 be added, twice the sum will be 28?

Let x denote the number.

Then $x+8$ is the number with 8 added, and $2(x+8)$ is twice the sum.

But we are told that twice this sum is 28.

Hence, $$2(x+8)=28;$$

$$\therefore\ 2x+16=28.$$

Transposing, $$2x=12,$$

$$x=6.$$

Ex. 3. Two rectangular boards are equal in area; the breadth of one is 18 inches, and that of the other 16 inches, the difference of their lengths is 4 inches. Find the length of each and their common area.

Let x denote the length of one board.

Then $x+4$ will denote the length of the other.

The areas of the two boards are respectively $18x$ and $16(x+4)$ (p. 231). But these areas are equal.

$$\therefore\ 18x=16(x+4),$$

or $$18x=16x+64.$$

Transposing, $$2x=64,$$

$$\therefore\ x=32, \text{ and } x+4=36.$$

Hence, length of one board is 32 inches.

And ,, other ,, 36 ,,

$\therefore$ Common area $=18\times32=16\times36=576$ square inches.

Ex. 4. A rectangle is 6 feet long; if it were 1 foot wider its area would be 30 square feet. Find the width.

Let x denote the width in feet.

Then $x+1$ is the width when one foot wider.

The area is $6(x+1)$, but the area is 30 square feet.

$$\therefore\ 6(x+1)=30,$$

or $$6x+6=30.$$

Transposing, $$6x=24,\ \therefore\ x=4.$$

Ex. 5. The age of the eldest of three children is equal to the sum of the ages of the other two, the ages of these two being in the ratio of two to three; in ten years the age of the eldest will be five years more than half the sum of the ages of the other two. Find their present ages.

Let $5x$, $3x$, and $2x$ be their present ages.

Then in ten years the age of the eldest will be $5x+10$.

But $$5x+10=\tfrac{1}{2}(2x+3x+20)+5;$$
$$\therefore\ 5x+10=\tfrac{1}{2}(5x+20)+5,$$
$$\therefore\ \tfrac{5}{2}x=5,\ \text{or } x=2.$$

Hence the ages are 10, 6, and 4.

Ex. 6. The diameters of two pulleys are as 2 to 3, and the sum of the diameters is 30 inches. Find the diameter of each.

Let $2x$ and $3x$ denote the diameters.

$$\text{Then } 2x+3x=30,\ \therefore\ x=6.$$

Hence the diameters are 12 inches and 18 inches.

EXERCISES. XXXIV.

1. What number is that to which, if 20 be added, the sum is equal to three times the required number?

2. *A* can do a piece of work in 8 days, *B* in 10 days, and *C* in 24 days; in what time will they finish it together?

3. What is the distinction in meaning between $a-(b-c)$ and $(a-b)+c$? Prove that they are equal.

4. What are the axioms on which the solution of a simple equation depends? Illustrate by the solution of
$$\frac{2x-1}{3}-x=\frac{3-x}{4}-\frac{3}{2}.$$

5. *A* is twice as old as *B*; twenty years ago he was three times as old. Find their ages.

6. Divide 24 into two parts, so that one part shall be seven times the other part.

7. Find two numbers, such that their sum is 51 and their difference is 15.

8. Find the time between 2 and 3 o'clock when the hands of a watch are (*a*) together, (*b*) opposite, (*c*) at right angles to each other.

9. Distinguish between an equation and an identity. Make an example of each. What value of c makes $(x-2)^2-(x-1)(x-3)=c$ an identity? Can any value of c make it an equation?

10. Divide a yard into two parts, such that half of one part with 22 inches may be double the other part.

11. Find two numbers whose sum is 39, and whose difference equals a third part of the greater.

12. The circumference of a large wheel is $3\frac{1}{2}$ times that of a smaller wheel; if the small wheel makes 10 turns more than the large wheel in running 21 yards, find the circumference of each wheel.

13. A market-woman spent 10s. 10d. in buying eggs, some at two a penny, others at three for twopence; she sold them for a sovereign, thereby gaining a halfpenny on each egg: how many of each kind did she buy?

14. Any sum of £23. 14s. is to be divided between A, B, and C; if B gets 20 per cent. more than A, and 25 per cent. more than C, how much does each get?

15. A man has 1000 apples for sale; at first he sells so as to gain at the rate of 50 per cent. on the cost price; when he has done this for a time the sale falls off, so he sells the remainder for what he can get, and finds that by doing so he loses at the rate of 10 per cent.; if his total gain is at the rate of 29 per cent., how many apples did he sell for what he could get?

16. Of two squares of carpet one measures 44 feet more round than the other, and 187 square feet more in area. What are their sizes?

17. A herd of 125 cattle is sold for £2575. There were half as many oxen again as there were cows, and the oxen fetched altogether £25 more than the cows. What was the price of each ox and of each cow?

18. A farmer has to pay rates amounting to 1s. 6d. an acre, but his landlord allows him 10 per cent. off his rent, and this allowance is £5 more than the rates. If the rates were to increase so as to amount to 2s. 6d. an acre, they would just be met by an allowance of 15 per cent. off the rent. Find the number of acres in the farm and the total rent.

Simultaneous Equations.—If an equation contains two unknown quantities, then by giving definite values to one of the unknown quantities, a corresponding series of values can be obtained for the other.

Ex. 1. Solve $3x - 5y = 6$.

By transposition, $3x = 5y + 6$; and giving values 1, 2, 3, etc. to y, we obtain a corresponding series of values of x.

If $y = 1$, then $3x = 11$; $\therefore\ x = \frac{11}{3}$.

$y = 2$, then $3x = 16$; $\therefore\ x = \frac{16}{3}$.

Proceeding in this manner, a table of values could be arranged as follows:

x	$\frac{11}{3}$	$\frac{16}{3}$	7	$\frac{26}{3}$	....
y	1	2	3	4	....

Thus, for any assigned value of y a corresponding value of x can be obtained.

In a similar manner, if values are assigned to x, corresponding values of y can be found.

If, now, we have a second equation $4x+3y=37$, as before, by giving any assigned value to either x or y, a corresponding value of the other unknown is obtained, and a table of corresponding values of x and y could be tabulated, as in the preceding case; comparing the two sets of values so obtained, it will be found that only one pair of values of x and y will satisfy both equations at once, the two values being $x=7$, $y=3$.

Equations such as

$$3x-5y=6,$$
$$4x+3y=37,$$

which are satisfied by the same values of the unknown quantities, are called *simultaneous equations.*

To find two unknown quantities, we must have two distinct and possible equations.

Ex. 2. $4x+3y=37$, and $12x+9y=111$.

These form two equations, but they are not distinct, as the second can be obtained from the first by multiplying by 3.

Again, $3x-5y=6$ and $6x-10y=13$ are two apparently impossible simultaneous equations, because, by multiplying the former by two, the equation becomes $6x-10y=12$, but from the second equation, $6x-10y=13$. Hence simultaneous values of x and y satisfying the two equations cannot be found. See Chap. xx.

To solve simultaneous equations, we require as many distinct and independent equations as there are unknowns to be found, *i.e.* if two unknowns have to be determined, two distinct equations are required; if three unknowns, three equations, and so on.

In the solution of a simultaneous equation containing two unknown quantities, there are two general methods by which their values may be obtained. The first is by multiplication or division to make the coefficients of one of the unknowns the same in the two equations; then by addition or subtraction, we can eliminate one unknown. This leaves an equation containing only one unknown, the value of which can be found in the usual manner.

The other method is to find the value of one unknown in terms of the other unknown in one of the equations, and then to substitute the value so found in the other equation.

Ex. 3.
$$3x-5y=6, \quad \text{(i)}$$
$$4x+3y=37. \quad \text{(ii)}$$

To apply the *first method*, multiply (i) by 3 and (ii) by 5.

$$\therefore \quad 9x-15y=18,$$
$$\underline{20x+15y=185.}$$

By addition,
$$29x \qquad =203;$$
$$\therefore \quad x=\frac{203}{29}=7.$$

Substitute this value in (i);
$$\therefore \quad 21-5y=6;$$

or
$$5y=21-6=15; \quad \therefore \quad y=3.$$

On substituting these values of x and y in the given equations, the equations are satisfied; the values obtained should always be substituted in this way to ensure accuracy.

By the second method:

From (i)
$$3x=6+5y;$$

or
$$x=\frac{6+5y}{3};$$
$$\therefore \quad 4x=\frac{24+20y}{3}.$$

Substitute this value in (ii);
$$\therefore \quad \frac{24+20y}{3}+3y=37.$$

Multiply both sides of the equation by 3,

then
$$24+20y+9y=111.$$

Hence,
$$29y=111-24=87;$$
$$\therefore \quad y=\frac{87}{29}=3.$$

Having found the value of y, then by substitution in (i) or (ii), the value of x is as before readily obtained.

The unknown quantities to be found from a simultaneous equation, are not necessarily expressed in experimental work as x and y terms, but p and v, p and t, A and B, E and W, etc., may be used instead.

It is often necessary and desirable to express the relation between two variable quantities by means of a formula or equation. The method by which the experimental results are obtained and plotted are explained in Chap. xx. on Graphics, but practice in solving a simultaneous equation is necessary before any such law can be determined.

Ex. 4. If the law of a machine is given by

$$B = aA + c,$$

and it is found that when B is 18, A is 8; and when B is 46, A is 16, find a and c.

Substituting the given values in the given equation

$$18 = 8a + c \quad \text{(ii)}$$
$$46 = 16a + c \quad \text{(iii)}$$
$$28 = 8a \quad \text{by subtracting (ii) from (iii).}$$

Hence $8a = 28$;

$$\therefore \ a = \tfrac{28}{8} = \tfrac{14}{4} = 3{\cdot}5.$$

Substituting this value in (ii),

$$c = 18 - (8 \times 3{\cdot}5)$$
$$= 18 - 28 = -10.$$

Hence the required law is, $B = 3{\cdot}5A - 10$.

Ex. 5. In the same machine if the relation between F and B is given by $F = aB + c$, and it is found when $F = 110$, then $B = 8$; and when F is 210, then $B = 16$.

We have

$$110 = 8a + c$$
$$210 = 16a + c$$
$$100 = 8a$$

$$\therefore \ a = \tfrac{100}{8} = 12{\cdot}5.$$

Hence $c = 110 - (8 \times 12{\cdot}5) = 110 - 100 = 10$.

And the required law is $F = 12{\cdot}5B + 10$.

As the solution of a simultaneous equation is of the utmost importance a few miscellaneous worked examples are added here:

Let

$$6x + 3y = 33, \quad \text{(i)}$$
$$13x - 4y = 19. \quad \text{(ii)}$$

Multiplying (i) by 4, we get $24x + 12y = 132$

,, (ii) by 3, $39x - 12y = 57$

By addition $63x = 189$

$$\therefore \ x = \tfrac{189}{63} = 3,$$

and by substitution in (i), $y = 5$.

From the preceding examples the student will have seen that in solving two simultaneous equations the object is to determine from the two given equations a value of one of the unknowns. This process is called **elimination.** Using the value so obtained we proceed to find the other. The artifices which may with advantage be employed in solving equations can only be obtained by practice.

The general methods previously explained may usually be employed. These methods are also made use of when more than two unknowns have to be found.

Ex. 6. Solve

$$x+y+z=53 \quad\ldots\ldots \text{(i)}$$

$$x+2y+3z=105 \quad\ldots\ldots \text{(ii)}$$

$$x+3y+4z=134 \quad\ldots\ldots \text{(iii)}$$

Subtract (i) from (ii), $\therefore\ y+2z=52$ ...... (iv)

,, (ii) from (iii), $y+z=29$ ...... (v)

By subtracting (v) from (iv), $z=23$

Substitute this value for z in (v),

$$\therefore\ y+23=29;$$

or $y=29-23=6.$

Again substituting for y and z in (i),

$$x+6+23=53;$$

$$\therefore\ x=53-29=24.$$

Hence the values are

$$\left.\begin{aligned} x&=24,\\ y&=6,\\ z&=23. \end{aligned}\right\}$$

Ex. 7. Solve

$$x+y+z=90 \quad\ldots\ldots \text{(i)}$$

$$2x+40=3y+20 \quad\ldots\ldots \text{(ii)}$$

$$2x+40=4z+10 \quad\ldots\ldots \text{(iii)}$$

Multiplying (i) by 2, $2x+2y+2z=180$ ...... (iv)

$2x-3y=-20$ ...... (ii)

Subtracting (ii) from (iv), $5y+2z=200$ ...... (v)

Subtracting (iii) from (iv), $2y+6z=210$ ...... (vi)

Multiply (v) by (3), $\therefore\ 15y+6z=600$

Subtract (vi), $2y+6z=210$

$$13y=390,\quad y=30.$$

From (v) $2z=200-150;$

$$\therefore\ z=25.$$

From (i) $x=90-30-25=35.$

If the known quantities are represented by the letters a, b, c, d, the solution is effected in the same manner.

Ex. 8. Solve $ax+by=c$ .... (i)

$bx+ay=d$ ....(ii)

Multiplying (i) by b, we get $abx+b^2y=bc$

,, (ii) by a, we get $\underline{abx+a^2y=ad}$

By subtraction, $b^2y-a^2y=bc-ad$

or $y(b^2-a^2)=bc-ad$;

$$\therefore\ y=\frac{bc-ad}{b^2-a^2}.$$

To obtain x we may either substitute for y, or proceed to eliminate y from (i) and (ii).

Thus multiplying (i) by a, $a^2x+aby=ac$

,, (ii) by b, $\underline{b^2x+aby=bd}$

Subtracting the upper line from lower, $(b^2-a^2)x=bd-ac$;

$$\therefore\ x=\frac{bd-ac}{b^2-a^2}.$$

EXERCISES. XXXV.

Solve the equations:

1. $\left.\begin{aligned}3x+\frac{9y}{2}&=42.\\ \frac{3x}{5}+4y&=27.\end{aligned}\right\}$

2. $\left.\begin{aligned}9x+5y&=65.\\ 7x-\frac{5y}{2}&=25.\end{aligned}\right\}$

3. $\left.\begin{aligned}\frac{x}{5}-\frac{y}{4}&=1.\\ 3x-4y&=10.\end{aligned}\right\}$

4. $2x-15y=3x-24y=1.$

5. $\left.\begin{aligned}7x+3y&=10.\\ 35x-6y&=1.\end{aligned}\right\}$

6. $\left.\begin{aligned}6x-12y&=1.\\ 8x+9y&=18.\end{aligned}\right\}$

7. $\left.\begin{aligned}5x+2y&=3\tfrac{1}{3}.\\ 6x-9y&=4.\end{aligned}\right\}$

8. $\left.\begin{aligned}5x-2y&=5.\\ 6y-2x&=11.\end{aligned}\right\}$

9. $\left.\begin{aligned}0{\cdot}5x+0{\cdot}07y&=0{\cdot}93.\\ 0{\cdot}03x-0{\cdot}4y&=0{\cdot}46.\end{aligned}\right\}$

10. $ax+by=bx-ay=c.$

11. $\begin{aligned}2x+3y+4z&=16.\\ 3x+2y-5z&=8.\\ 5x-6y+3z&=6.\end{aligned}$

12. $\begin{aligned}3x+2y-z&=20.\\ 2x+3y+6z&=70.\\ x-y+6z&=41.\end{aligned}$

Solve the equations:

13. $\frac{x}{2}+\frac{y}{3}=12-\frac{z}{6}$.
$\frac{y}{2}+\frac{z}{3}=8+\frac{x}{6}$.
$\frac{x}{3}+\frac{z}{2}=10$.

14. $2x+3y+4z=29$.
$3x+2y+5z=32$.
$4x+3y+2z=25$.

15. $\frac{x}{2}+\frac{y}{3}+\frac{z}{4}=62$.
$\frac{x}{3}+\frac{y}{4}+\frac{z}{5}=47$.
$\frac{x}{4}+\frac{y}{5}+\frac{z}{6}=38$.

16. $x-y+z=5$.
$3x+4y-5z=13$.
$x+\frac{1}{2}y+\frac{1}{3}z=14$.

17. $\frac{x}{7}-\frac{y}{5}=1$.
$\frac{x}{5}+\frac{y}{7}=\frac{7}{5}$.

18. $\frac{x}{2}-y=\frac{7}{2}$.
$\frac{x}{25}+\frac{y}{2}=-\frac{3}{10}$.

19. $\frac{7x+3}{5}=\frac{5y-7}{4}=\frac{2x+3y-5}{3}$.

20. $0\cdot5x-0\cdot6y=1\cdot38$.
$\frac{x}{3}+\frac{y}{0\cdot5}=2$.

21. $\frac{x-y}{2}-\frac{3x-\frac{1}{2}y}{5}=\frac{1}{2}$.
$\frac{x}{3}-\frac{y}{2}=2$.

22. $2x-7y=0\cdot2$.
$0\cdot2x+0\cdot15y=1\cdot72$.

23. $3\cdot4x-0\cdot02y=0\cdot01$.
$x+0\cdot2y=0\cdot6$.

24. $\frac{3y}{4}-\frac{2x}{5}=\frac{13}{5}$.
$\frac{x}{4}+\frac{y}{5}=\frac{19}{12}$.

25. $\frac{x}{15}+\frac{y}{12}=\frac{x}{3}-\frac{y}{4}=1$.

26. $\left.\begin{aligned}(x+y)^2-(x-y)^2&=352.\\ x(y+5)&=143.\end{aligned}\right\}$

27. $2x+3y=3x+2y=25$.

28. $5x+2y=3\frac{1}{3}$.
$6x-9y=4$.

29. $5x-2y=5$.
$6y-2x=11$.

30. Find the value of $x^3-8y^3+29z^3+18xyz$ when $2y=x+3z$, and $z=5$.

31. Solve the equations: $\left.\begin{aligned}2x+3y&=5.\\ 10x-6y&=11.\end{aligned}\right\}$

Problems leading to simultaneous equations.—We have already found that to solve a simultaneous equation it is essential to have as many independent equations as there are unknown quantities to determine.

Therefore in the solution of any problem which produces a simultaneous equation it is necessary in the statement of the question that there should be as many equations involved as there are unknowns to be determined.

Ex. 1. If 3 be added to the numerator of a certain fraction, its value will be $\frac{1}{3}$, and if 1 be subtracted from the denominator its value will be $\frac{1}{5}$. What is the fraction?

Let x be the numerator, and y the denominator of the fraction.

Add 3 to the numerator, then $\frac{x+3}{y}=\frac{1}{3}$.

Subtract 1 from the denominator, and $\frac{x}{y-1}=\frac{1}{5}$.

$$\therefore \quad \frac{x+3}{y}=\frac{1}{3}, \text{ and } \frac{x}{y-1}=\frac{1}{5}.$$

$$\therefore \quad 3x+9=y \quad \text{(i)}$$

and $$5x=y-1. \quad \text{(ii)}$$

Transposing we get $$y-3x=9 \quad \text{(iii)}$$

$$y-5x=1 \quad \text{(iv)}$$

Subtracting (iv) from (iii) $2x=8, \quad \therefore \quad x=4.$

Substituting this value of x in (iii)

$$y-12=9, \quad \therefore \quad y=21.$$

Hence the fraction is $\frac{4}{21}$.

EXERCISES. XXXVI.

1. Find the fraction which becomes equal to $\frac{1}{2}$ if its denominator be decreased by 6, and becomes equal to $\frac{3}{4}$ if its numerator and denominator be increased by 124.

2. A bill of £13. 12s. 6d. was paid with 40 coins, some of them half-crowns and the others half-sovereigns: find the number used of each kind.

3. If 10 yards of silk and 7 yards of satin cost £5. 6s. 4d.; and if 3 yards of the satin cost as much as 4 yards of the silk, find the price of a yard of each.

4. A certain number, consisting of two digits, exceeds four times the sum of its digits by 3; if the number be increased by 18, the

result is the same as if the number formed by reversing the digits were diminished by 18; what is the number?

5. Find the fraction which is equal to $\frac{3}{7}$ when 10 is added to its numerator, and which is equal to $\frac{1}{3}$ when 4 is subtracted from its denominator.

6. The numerator of a certain fraction is 4 less than the denominator; if 10 be subtracted from the numerator, or if 30 be added to the denominator, the resulting fractions would be equal: find the original fraction.

7. (i) The expression $ax^2+bx-30$ is equal to 240 when $x=5$, and is equal to 100 when $x=-2$; find the values of a and b.

(ii) If Q varies as $H^{\frac{5}{2}}$, and Q is $7{\cdot}26$ when H is $1{\cdot}5$, find H when Q is $5{\cdot}68$.

8. A quantity x equals the sum of two quantities, one of which varies as t the other as t^2; when t equals 3, x equals 159, and when t equals 5, x equals 425; find the value of x when t equals 10.

MISCELLANEOUS EXERCISES. XXXVII.

Solve the equations:

1. (i) $\dfrac{x+3}{7}-\dfrac{2x-5}{3}=\dfrac{3x-5}{5}-25.$ (ii) $\frac{2}{7}(x-5)+\frac{5}{8}(x-4)=8\frac{5}{8}.$

2. (i) $x^2-11x-26=0.$ (ii) $\dfrac{x}{8}+\dfrac{y}{3}=15;\ \dfrac{x}{4}-\dfrac{y}{5}=4.$

3. (i) $\frac{1}{12}x-\frac{1}{8}(8-x)-\frac{1}{4}(1+x)+\frac{7}{4}=0.$

(ii) $\dfrac{1}{x-1}-\dfrac{2}{x+7}=\dfrac{1}{7(x-1)}.$

4. A contribution of £125 was raised by A, B, and C; A gave twice as much as B, and B gave £15 more than C; what did each give?

5. A man bought some tea of two sorts, 3 lbs. of the first costing as much as 4 lbs. of the second, while 9 lbs. of the first and 8 lbs. of the second together cost £2. 5s.: what was the price of each per lb.?

Solve the equations:

6. (i) $2x-\dfrac{5x-4}{6}=7-\dfrac{1-2x}{5}.$ (ii) $\dfrac{x}{8}-\dfrac{2x-1}{5}=\dfrac{3x-4}{15}-\dfrac{x}{120}.$

7. (i) $\dfrac{7x}{3}+\dfrac{4y}{9}=15,$ (ii) $4x=\dfrac{3y}{5}+1\frac{1}{5}.$

8. (i) $\frac{1}{5}(2x+11)-\frac{1}{2}(5-6x)=7x+1\frac{1}{2}.$

(ii) $4\left(x-\frac{3}{8}\right)-3\frac{1}{5}\left(\frac{1}{2}x-1\right)=5\frac{9}{10}.$

9. (i) $0{\cdot}5x+0{\cdot}6x-0{\cdot}8=0{\cdot}75x+0{\cdot}25.$

(ii) $\dfrac{x-1}{4}-\dfrac{1}{8}\left(\dfrac{x-5}{4}-\dfrac{14-2x}{5}\right)=\dfrac{x-9}{2}-\dfrac{7}{8}.$

10. $\left.\begin{aligned} 7x-2y&=14+\tfrac{1}{2}x, \\ 7y-2x&=32+\tfrac{1}{3}y. \end{aligned}\right\}$

11. (i) $\dfrac{3x-7}{10}-\dfrac{7x-33}{25}+\dfrac{4x+9}{15}=6\tfrac{2}{3}$; (ii) $\left.\begin{aligned} 6x+4y&=61, \\ 8x-5y&=\ \ 9. \end{aligned}\right\}$

12. $\dfrac{x}{2}+\dfrac{\dfrac{3x-1}{4}-\dfrac{2x-3}{3}}{\dfrac{x-1}{2}}=\dfrac{3(x^2+2)}{2(3x-2)}.$

13. What values of x will make the expression $x^2+4{\cdot}8x+2{\cdot}87$ have the value zero?

14. Divide £490 among *A*, *B*, and *C*, so that *B* shall have £2 more than *A*, and *C* as many times *B*'s share as there are shillings in *A*'s share.

15. In a certain proper fraction the difference between the numerator and the denominator is 8, and if each be diminished by 7, the fraction becomes equal to $\frac{4}{5}$. Find the fraction.

16. An army in a defeat loses one-sixth of its number in killed and wounded, and 4,000 prisoners. It is reinforced by 3,000 men, but retreats losing a fourth of its number in doing so. There remains 18,000 men. What was the original force?

17. The expression $ax^2+bx-30$ is equal to 240, when x equals 5, and is equal to 100, when x equals -2; find the values of a and b.

18. The electrical resistance of a wire of given material varies directly as the length and inversely as the square of the diameter. Find the length and diameter of a wire which is to have three-fourths of the electrical resistance, and one-third of the weight of a wire of the same material 50 ft. long and 0·03 in. diameter.

Summary.

Identity.—An equality involving only an algebraic operation between two quantities is called an identity.

Simple Equations.—A statement of equality between two algebraic expressions is called an equation, and is called a simple equation when it involves only the first power of the unknown quantity.

In an equation the results are still equal if corresponding operations (addition, subtraction, multiplication, and division) are performed on both sides of the equation.

Simultaneous Equations.—In an equation which contains two unknown quantities x and y, an indefinite number of pairs of values may be found which will satisfy the equation; but if a second equation be given, only one pair can be found which will simultaneously satisfy both equations.

CHAPTER IX.

GRAPHIC METHODS. SCALES AND THEIR USE. GRAPHIC ARITHMETIC.

Graphic methods.—In the calculations which have to be made continually by the practical man, there is always the risk, even when great care is exercised, of errors finding their way into the work. Hence some trustworthy and independent method by which the calculations can be checked is desirable. Such a check is furnished by graphic methods, in which drawing instruments can be used for all, or nearly all, the problems with which the practical man has to deal.

With care and the use of good scales, results are easily and readily obtained with considerable accuracy.

Scales and their use.—The majority of the problems considered are supposed to be solved by the process known as drawing to scale. In making a drawing of any large object, such as a building, it would be inconvenient, if not impossible, to make it as large as the object itself. In other words, to draw it *full size* is impossible, but if a drawing were made in which every foot length of the building were represented on the drawing by a length of half an inch, the drawing would be said to be drawn to a scale of $\frac{1}{2}$ inch to a foot, or to a scale of $\frac{1}{24}$.

By means of a suitable scale any required dimension could be obtained as readily as if the drawing were made full size. In a similar manner, if the drawing were made so that every length of 3 inches on the drawing represented an actual length of 12 inches, the scale would be said to be $\frac{1}{4}$. This fraction of $\frac{1}{24}$, or $\frac{1}{4}$, etc., is called the **representative fraction of the scale.**

Hence, *Representative fraction of a scale*

$$=\frac{\text{number of units in any line on the drawing}}{\text{number of units the line represents}}.$$

The term representative fraction is not always used, but, more shortly, the drawing is said to be made to a scale of $\frac{1}{4}$ or $\frac{1}{24}$.

Scales of boxwood or ivory are readily obtainable; the former are cheaper than the latter, and the student should possess at least one good boxwood scale about 12 inches long. What is called an *open* divided scale will be found most useful. These can be obtained with the following scales: 1″, $\frac{1}{2}$″, $\frac{1}{4}$″, and $\frac{1}{8}$″ on one side, all divided in eighths. The same scales in tenths are found on the obverse side. Such a scale is shown in Fig. 34. These scales are divided up to the edge, which is made thin, as shown in the sections *a* and *b*, and so allows dimensions to be marked off direct from the scale with a fine-pointed pencil or pricker.

It is not advisable to use compasses or dividers, if it can be avoided, when transferring dimensions from scales. The frequent use of dividers on scales soon wears away the divisions on the scale, and renders them useless for accurate measurements.

Division of a line.—Given any line *AB* (Fig. 35), to divide into a number of equal parts is comparatively an easy task when an even number of parts are given, such as 2, 4, 8, etc. In such a case the line would be bisected by using the dividers, each part so obtained again bisected, etc. When an odd number of parts are required, such as 5, etc., a length may be taken representing about one-fifth of *AB* (Fig. 35). This, on trial, may prove to be slightly longer or shorter than the necessary length. By alteration

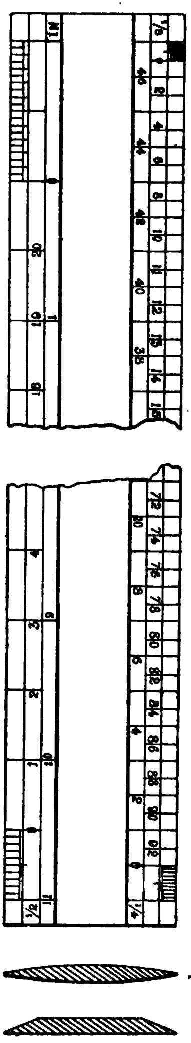


Fig. 34.—Open divided boxwood scale.

of the dimension in the required direction, and by repeated trials, a length is ultimately found which is exactly one-fifth. Much time and labour may be spent in this way. Another and better method is to set off a line AC (Fig. 35) at any convenient angle to AB, and to mark off *any* five equal lengths along AC from A to 5, and join 5 to B. If lines are drawn through the successive points 1, 2, 3, 4, parallel to the line $5B$, then AB will be divided into the required number of equal parts.

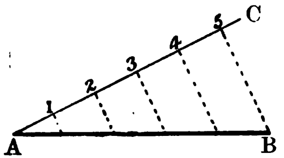


FIG. 35.—Division of a line.

Ex. 1. Given a line AB representing 7 units, to find the length of unit adopted.

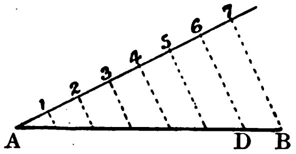


FIG. 36.—To find the unit of length, when a line AB represents 7 units.

This is similar to the preceding, and obviously the unit is obtained by measuring AB, and dividing the number of units so obtained by 7. A better method is to set off 7 equal units along a line, inclined at any angle to AB, as shown in Fig. 36. Join 7 to B, and from division 6 draw a line parallel to $7B$, intersecting AB at D.

Then DB is the required unit.

Making a scale.—Although scales are easily obtained, it sometimes happens that a particular scale is not to hand, and therefore a scale has to be made. In making such a scale it is advisable to use either good cartridge paper or thin cardboard, so that when made it can be cut out and carefully laid aside for use on a future occasion.

Let the scale required be $\frac{1}{6}$, or 2 inches to 1 foot. Draw two lines about $\frac{1}{8}$ or $\frac{1}{10}$ of an inch apart, set off a length $A0$ of 2 inches (Fig. 37), and divide the length as accurately as possible into twelve equal parts as shown. These divisions, denoting inches, are numbered from 0 to A (Fig. 37). In addition, as many lengths 01, etc., indicating feet, as may be necessary are marked off in the opposite direction. When the scale is divided in this manner, it enables a dimension, including feet and inches, to be read off easily. One point of the dividers or

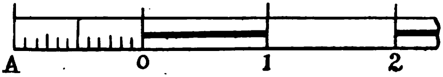


FIG. 87.—Making a scale.

compasses can be placed in one of the divisions denoting the number of feet required, and the other end opened until the space between the two points is the distance required.

Ex. 2. Construct scales of $\frac{1}{8}$, $\frac{1}{4}$, $\frac{1}{2}$, $\frac{1}{6}$, $\frac{1}{12}$, and $\frac{1}{24}$.

EXERCISES. XXXVIII.

1. A line 4 inches long denotes a scale of 20 yards. Deduce from it a scale of feet to read 1′ 0″, and show 70′ 0″.

2. A line $3\frac{3}{4}$ inches long represents a scale of 50 feet. Convert the given scale into a scale reading inches. State the representative fraction of this scale.

3. A line AB, $1\frac{1}{2}$ inches long, represents a length of 45 feet. Construct and figure correctly the scale to which AB is drawn.

4. Draw a scale, the representative fraction of which is $\frac{1}{360}$, reading yards.

5. Construct a scale of feet for a drawing on which 7 feet 6 inches are represented by 1·5 inches. The scale is to be correctly divided and figured.

6. A line AB, $3\frac{1}{2}$ inches long, represents a length of 35 feet. Construct and figure correctly the scale to which AB is drawn.

7. Construct and figure correctly a scale of ·8″ to 10′ 0″, capable of giving dimensions up to 90′ 0″.

8. Construct a scale of 3·5″ to 30′, to show feet, and correctly figured. Draw to the scale a triangle, its sides respectively 20′, 15′, and 12′ long.

Graphic Arithmetic.—By the use of suitable scales numbers may be represented by lines, and ordinary arithmetical results may be obtained by geometrical construction. The scale selected will vary according to the size of the quantity to be measured; when the quantities are small a comparatively large scale may be used, and the unit length may be 1″, $\frac{1}{2}$″, etc., which enables a fairly accurate estimate to be made. To represent larger quantities, the unit may be $\frac{1}{10}''$, $\frac{1}{100}''$, $\frac{1}{1000}''$, etc. Although a smaller scale is used for larger quantities, the percentage error, if care is exercised, is not greater. This use of lines is called *Graphic Arithmetic*, and the processes by which the various results are obtained are merely applications of the principles of proportion, as shown on p. 70.

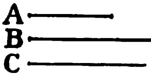


Fig. 38.

Addition.—*Ex.* 1. To determine by construction a line representing the sum of the three given lines *A*, *B*, and *C* (Fig. 38), we may proceed as follows:

Place the three lines *A*, *B*, and *C* end to end and in the same straight line; then the total length so obtained is the sum required.

Ex. 2. Find the sum of $5+3+4$.

Using a $\frac{1}{4}''$ scale for the unit, we mark off from any convenient point 5 quarter inches, next 3 units, and finally 4; if the scale be now applied to the line, the length will measure 12 units, which is the sum required.

Subtraction.—Having selected a suitable unit, draw a line of indefinite length; set off as many units as there are in the given quantity, and the result required is obtained if from the division denoting the last unit a line is set off in the reverse direction equal in length to the number to be subtracted.

Ex. 3. Subtract 5 from 12. Unit $\frac{1}{4}''$.

Draw any line *ad* (Fig. 39), set off $ad=12$ quarter inches; from *d* in the reverse direction set off *db* equal to 5 units, then *ab* will be found to measure 7 units on the scale.

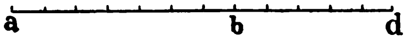


Fig. 39.- Subtraction.

Note.—Quantities to be added are called *positive quantities*, and are measured from a point to the right of that point; those to be subtracted are called *negative quantities*, and are, as shown, measured to the left.

Proportion.—When four quantities are proportional, we may write them as $A:B::C:D$ (p. 70), and given *A*, *B*, and *C*, we proceed to find the fourth proportional as follows:

Draw two lines at any convenient angle to each other. In Fig. 40 the lines are at right angles. Set off a distance $oa=A$

along the vertical line, and a distance $ob = B$ along the horizontal line. Join a to b. Set off the third quantity C along the vertical line, making $oc = C$; draw a line cd parallel to ab, and meeting ob produced at d, then $od = D$ is the fourth proportional, or answer required.

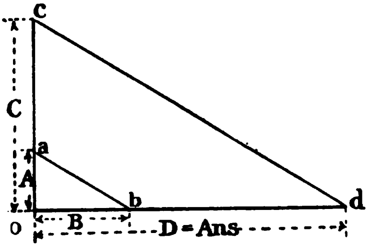


FIG. 40.—Proportion.

Denoting the fourth proportional by x, then $A : B :: C : x$, as on p. 70, or, multiplying extremes and means,

$$A \times x = B \times C; \quad \therefore \quad x = \frac{B \times C}{A}.$$

The result may be expressed as the equality of two ratios:

$$\frac{x}{B} = \frac{C}{A}.$$

In many cases the value of a complicated fraction can be found by proportion; the method employed may be shown by the following example.

Ex. 4. Find the value of $\frac{1\frac{3}{4} \times 2\frac{7}{8}}{1\frac{3}{8}}$.

In Fig. 41, on a convenient scale, ob is made $= 1\frac{3}{4}$, $oa = 1\frac{3}{8}$, and $oc = 2\frac{7}{8}$.

Join a to b, and from c draw cd parallel to ab, meeting ob produced at d.

Then od is the required result.

When measured, od will be found to be 3·7 units.

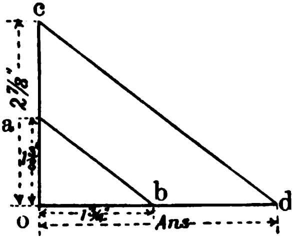


FIG. 41.—Simplification of a fraction.

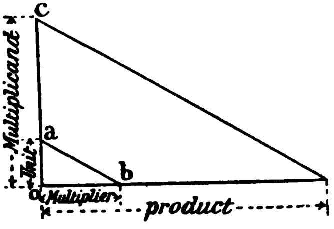


FIG. 42.—Multiplication.

Multiplication. — Given two quantities B and C, to find the product, or $B \times C$ (Fig. 42).

When four quantities form a proportion, we have seen that, when three terms are given, the fourth x is obtained by multiplying by two of the terms and dividing by the remaining one; that is,

$$x=\frac{B\times C}{A}.$$

If *the divisor be unity*, then $x\times 1=B\times C$.

Ex. 5. Find the product of 4·5 and 3·2.

$$\therefore\ x\times 1=4{\cdot}5\times 3{\cdot}2,\ \text{or}\ x=14{\cdot}4.$$

Draw two lines *od* and *oc* making any convenient angle with each other (in Fig. 43 this angle is 90°).

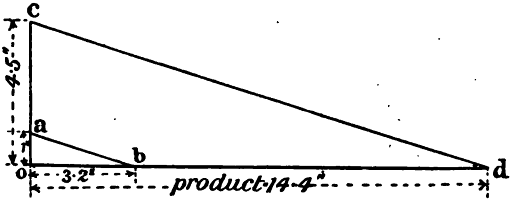


FIG. 43.

Along *oc* set off *oa* = unity, along *od* *ob* = 3·2 units, and *oc* vertically = 4·5 units.

Join *a* to *b* and draw *cd* parallel to *ab* and meeting the line *od* at *d*.

Then *od* will represent the product of *ob* and *oc*, and will be found to measure 14·4 units.

Ex. 6. Find the product of $4\frac{7}{8}$ and $3\frac{3}{8}$.

Division.—Again referring to proportion expressed as the equality of two ratios, we have

$$x=\frac{B\times C}{A}.$$

To divide any given quantity C by A, make B unity, then

$$x=\frac{C\times 1}{A}.$$

Set off oa = unity. Also *oc* the divisor and *od* the dividend (Fig. 44).

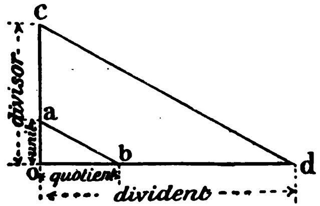


FIG. 44.—Division.

Join *c* to *d*, and from *a* draw a line *ab* parallel to *cd*, then *b* the point of intersection of *ab* with *od* determines a length *ob*, which, measured, gives the quotient required.

Ex. 7. Divide $4\frac{7}{8}$ by $3\frac{3}{8}$. Draw two lines at right angles, as shown in Fig. 45, set off lengths equal to $3\frac{3}{8}$ units, and $4\frac{7}{8}$.

Proceed as before to obtain the quotient which, when measured to scale, is found to be $1\frac{4}{9}$ units.

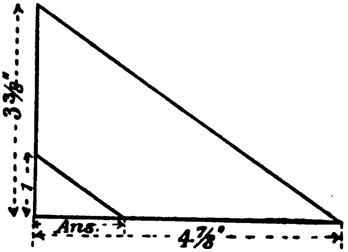


FIG. 45.

EXERCISES. XXXIX.

1. Determine by construction a line representing the value of

$$\frac{1\frac{1}{4}'' \times 2\frac{1}{2}''}{1\frac{3}{4}''}.$$

2. A line $1\frac{3}{4}''$ long represents the value of $\frac{2\frac{1}{4}'' + 1\frac{1}{4}''}{5\frac{1}{4}''}$. Determine and write down the length of the unit.

3. If a line $1\frac{3}{4}''$ long represents the unit, what numbers do lines $3\frac{1}{2}''$, $1\frac{1}{4}''$, and $1\frac{1}{2}''$ severally represent? Determine also a line representing

$$\frac{3\frac{1}{2}''}{1\frac{1}{4}'' \times 1\frac{1}{2}''}.$$

4. If a line $1''$ long is the unit of length, determine and write down the number of units of area in the rectangle, the adjacent sides of which are $2\frac{1}{2}''$ and $1\frac{3}{8}''$.

5. If a line $2\frac{1}{2}''$ represents the product of $1\frac{1}{2}'' \times \frac{3}{4}''$, what length of line will represent the product of $2\frac{1}{2}'' \times \frac{3}{4}''$?

6. If $1\frac{1}{2}''$ is the unit, determine and write down the length of line which will represent the product of $1\frac{1}{2}'' \times 2\frac{1}{2}''$.

7. What is meant by the mean proportional between two lines? Find the mean proportional between two lines respectively $1\frac{1}{4}''$ and $2''$ long. Divide a line $4\frac{1}{2}''$ in length into three consecutive parts to each other in the proportion of $1 : \frac{3}{4} : \frac{1}{2}$.

Involution.—Given a line L and a unit line oa, to find L^2, L^3, etc. Draw two lines od and of at right angles.

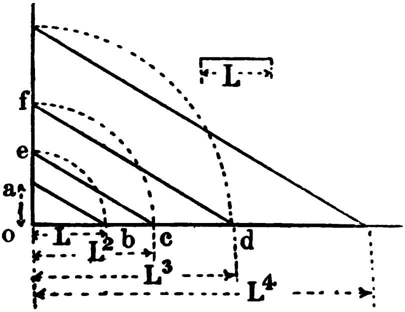


FIG. 46.—Involution.

Set off the given unit from o to a (Fig. 46), and along the horizontal line mark off $ob = L$. Join a to b, and with o as centre, radius ob, describe an arc of a circle cutting the vertical at e.

Draw ec parallel to ab, meeting ob produced at c. Then $oc = L^2$.

In a similar manner, with o as centre, radius oc, describe an arc of a circle cutting the vertical at f. Draw fd parallel to ab or ec meeting the horizontal line at d. Then $od = L^3$.

By proceeding in this manner L^4, L^5, etc., may be obtained.

Ex. 1. To obtain $(2\frac{7}{8})^2$, $(2\frac{7}{8})^3$, etc.

Using any convenient scale such as $\cdot 5'' = 1$.

Mark off oa = unity and $ob = 2\frac{7}{8}$, and proceed as before, so obtaining a figure similar to that shown in Fig. 47.

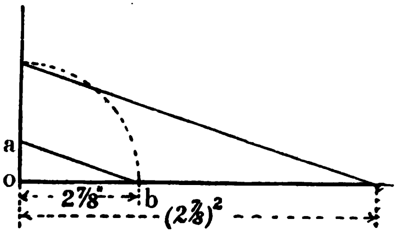


FIG. 47.

Square root.—The square root of a number is often required in practical calculations, and may be calculated as already explained in Arithmetic (p. 64), or obtained by means of the slide rule (p. 184), or by graphical construction, as follows:

Ex. 2. *Find the square root of* $4\frac{3}{4}$.

Using any convenient scale, mark off $ab = 4\frac{3}{4}$, and bc = unity on same scale.

On *ac* describe a semicircle, and at *b* draw *bd* perpendicular to *ac*, and meeting the semicircle in *d*.

Then *bd* is the *square root required.*

As already shown on page 70, *bd* is a mean proportional to the two quantities *ab* and *bc*.

The construction shown in Fig. 48 is that of finding *the mean proportional of the two numbers* $4\frac{3}{4}$ *and unity.*

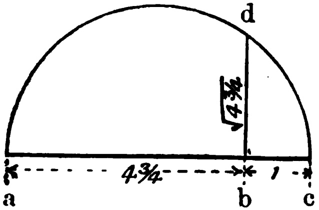


FIG. 48.—Square root.

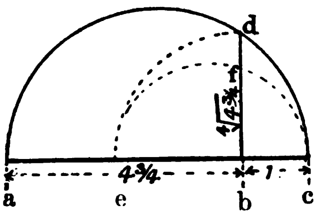


FIG. 49.

Ex. 3. *To obtain the fourth root of* $4\frac{3}{4}$ *or* $\sqrt[4]{4\frac{3}{4}}$.

Having obtained, as in the previous example, the square root *bd*, make *be* (Fig. 49) equal to *bd*. This is easily effected by using *b* as centre, *bd* as radius, and describing the arc *be*, meeting *ac* in *e*.

On *ec* describe a semicircle. Let *f* be the point of intersection of *bd* with the semicircle.

Then *bf* is the fourth root required.

In a similar manner the 8th, 16th, etc., roots can be obtained.

EXERCISES. XL.

1. A line 2·6″ long represents $4\frac{1}{2}$ units. Obtain lines representing 13, $\frac{10}{3}$, and $2\sqrt{6}$.

2. Supposing that the area of a circle is $\frac{22}{7}r^2$, where r is the radius; obtain by construction, and write down the length of a line representing the area of a circle 3″ in diameter. Unit 1″.

3. Find the line which represents the cube of a line $1\frac{1}{8}''$ long, to a unit of 1″.

4. Find the line which represents $\frac{5}{7}$, to a unit of $\frac{1}{2}$ inch.

5. Determine a line whose length shall represent $\sqrt{\frac{7}{2}}$, taking $\frac{3}{4}''$ as the unit.

6. If a line 2 inches in length represents the fraction $\frac{7}{4}$, determine the unit.

7. If the area of an equilateral triangle of $2\frac{1}{4}''$ side is represented by a length $3\frac{1}{4}''$, what is the unit?

8. Draw a curve which will give the squares of all quantities from 0 to 6, taking $\frac{1}{8}''$ as the unit.

Taking $\frac{1}{4}$ inch as unit, obtain lines representing

$$15, \sqrt{15}, \text{ and } \frac{5}{\sqrt{15}}.$$

9. Find by construction the square roots of 2, 3, 5, 7, 10; also the eighth root of 6561.

Right-angled triangles.—*In a right-angled triangle the square on the hypothenuse is equal to the sum of the squares on the other two sides.*

In Fig. 50 the line AB, on which the triangle appears to stand, is for convenience called the base of the triangle, and the line BC perpendicular to this, the height. The sloping side AC, which is the longest of the three, is called the hypothenuse.

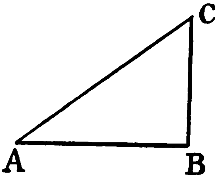


Fig. 50.—Right-angled triangle.

(i) *Verification by construction.*—Draw a right-angled triangle ABC, making on any convenient scale the base AB 30 units and the perpendicular BC 40 units.

Join A to C; if AC be measured on the same scale, its length will be found to be 50 units.

(ii) *By calculation.*—$AC^2 = AB^2 + BC^2$

$$= 30^2 + 40^2 = 2500; \quad \therefore \quad AC = \sqrt{2500} = 50.$$

This property of a triangle that, when the three sides are proportional to the numbers 3, 4, and 5, the angle opposite the longest length is a right angle, is made use of by builders and others to set out perpendicular lines, or "one line square with another."

The important relation between the three sides can easily be shown by using a piece of cartridge paper or thin cardboard, as follows:

Denoting the two sides at right angles by a and c, and the hypothenuse by b.

Draw a square *EBGD* (Fig. 51), having each of its sides equal to the sum of $a+c$, and a second square *HFMN* (Fig. 52) of the same size.

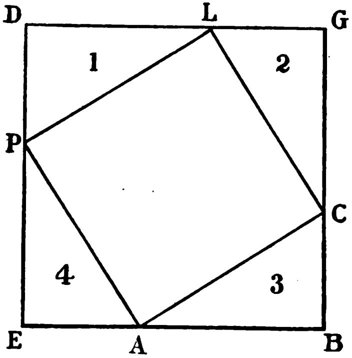


FIG. 51.

The two squares which are shaded are the squares on the sides a and b.

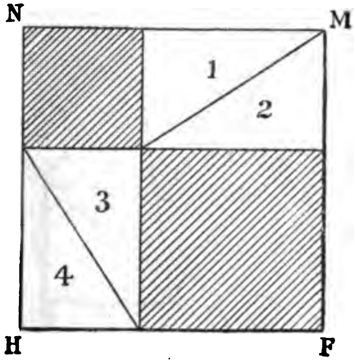


FIG. 52.

Along the four sides of the square *EBGD* mark off a length equal to c from *B* to *A*, *G* to *C*, *D* to *L*, and *E* to *P*.

Join *PL*, *LC*, *CA*, and *AP*.

Then $ACLP$ is an inscribed square, its side b units in length. The triangles 1, 2, 3, 4 in the square $EBGD$ have their sides respectively equal to those marked in the same manner in the square $HFMN$, and the four triangles in the one square are equal in all respects to those in the other.

Hence, if we imagine these four triangles removed it follows that the inscribed square, its side b units in length, in the square $EBGD$ is equal to the two squares in $HFMN$.

Thus in a right angled triangle, its perpendicular sides a and c and hypothenuse b,

$$b^2 = a^2 + c^2.$$

From this relation when any two of the three sides are given, the third can be found.

Ex. 1. If the hypothenuse be 17 and the base 8, then

$$\text{height} = \sqrt{17^2 - 8^2} = 15.$$

Since $$17^2 - 8^2 = (17+8)(17-8) = 25 \times 9\,;$$

$$\therefore\ \sqrt{25 \times 9} = 5 \times 3 = 15.$$

By successive applications of a right-angled triangle the square roots of the first n natural numbers (1, 2, 3...n) can be obtained.

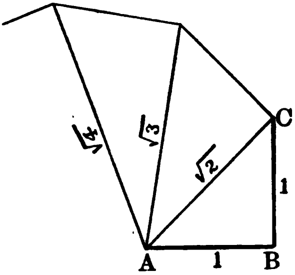


FIG. 53.

Ex. 2. Draw a line $AB = 1$ (Fig. 53), also BC perpendicular to AB and equal to 1. Join AC, then $AC = \sqrt{2}$. Draw CD perpendicular to AC and equal to 1. Join CD, then $AD = \sqrt{3}$.

Proceed in a similar manner to obtain the remaining values.

EXERCISES. XLI.

The following examples may be solved graphically:

1. The hypothenuse of a right-angled triangle is 177, and the base 141: find the remaining side.

2. If the two perpendicular sides of a right-angled triangle are 27 and 36, find the hypothenuse.

3. The base is 5·7 in., and the perpendicular 1 ft. $5\frac{3}{8}$ in.: find the hypothenuse.

4. One end of a ladder rests against a wall at a distance of 35 ft. from the ground; the bottom of the ladder is at a distance of 4 yards from the wall: find the length of the ladder.

5. Given the base of a right-angled triangle 12, and the perpendicular 16, find the length of the hypothenuse.

6. Given the base 182·7, and the hypothenuse 288, find the perpendicular.

7. Given the hypothenuse 560·14 ft., and the perpendicular 454·79 ft., find the base.

8. The sides of a rectangular field are respectively 563 and 369 links: find the diagonal.

9. A ladder 40 ft long just reaches to the top of a wall; the bottom of the ladder is 6 ft. from the wall: find the height of the wall.

10. Find the side of a square field whose area is 15 acres.

11. The perimeter of a right-angled triangle is 24 ft., and its base is 8 ft.: find the other sides.

12. Find the hypothenuse of a right-angled triangle, one of its sides 39 in. and the area 84·5 sq. ft.

13. A field is in the form of a rectangle whose adjacent sides are 510 and 680 ft. respectively: find the length of a path from one corner to the opposite one.

Summary.

Representative fraction of a scale

$$= \frac{\text{number of units in any line on drawing}}{\text{the number of units the line represents}}$$

Right-angled triangles.—In a right-angled triangle the square on the hypothenuse, is equal to the sum of the squares on the other two sides.

CHAPTER X.

LABOUR-SAVING METHODS: LOGARITHMS. MULTIPLICATION, DIVISION, INVOLUTION AND EVOLUTION BY LOGARITHMS.

The use of Logarithms to facilitate calculations.—On account of the labour involved in the processes of multiplication, division, involution and evolution, many labour-saving devices are used by practical men. By such means many operations which would otherwise require considerable labour, and consequent risk of error, are performed readily, and in many cases almost mechanically.

Amongst the many forms of labour-saving contrivances we can only refer to a few. Perhaps the most common is that of a carefully compiled table of numbers by means of which, knowing the weight or price of a single article, the weight or price of any number can at once be ascertained. This gives rise to the so-called "**ready reckoners.**"

Also, in special cases where a large amount of multiplication, division, etc., has to be performed, one or other of the many forms of **calculating machines** may be used. These latter are far too expensive for general use. Hence, when it is necessary for any practical purpose to multiply or divide one set of numbers by another, the contracted methods shown on page 60 may be used. But multiplication and division as well as involution and evolution can be much more readily and easily performed by either *logarithms* or the *slide-rule.*

Logarithms and Logarithmic Tables.—As it is necessary that both the student and the practical man should be able to use logarithmic tables with ease and facility, and as they may be used readily even by those who are not acquainted with the manner in which they are usually calculated and tabulated, it is desirable before entering into any explanation as to the principles on which common and hyperbolic logarithms are calculated to consider how, by means of a table of logarithms, the troublesome and tedious arithmetical operations of multiplication, division, involution and evolution are readily and accurately performed. We shall also find that many problems which would be impossible by arithmetical processes are readily solved when logarithms are used. Again, in using a slide-rule a knowledge of the use of logarithms is of service, enabling the significance of its divisions to be far more intelligible than would otherwise be the case.

Logarithms of numbers consist of an integral part called the **index** or **characteristic**, and a decimal part called the **mantissa.** If the reader will refer to Table IV., he will find that opposite any given number from 10 to 99 four figures are placed. These four figures are called the **mantissa**, the **characteristic** has to be supplied when writing down the logarithm of any given number in a way to be presently described.

Where great accuracy is required, seven or more figures are to be found in the mantissa. Logarithmic tables of all numbers from 1 to 100000 have been calculated with seven figures in the mantissa, but for all practical purposes, and where only approximate calculations are required, such a table as that shown in Table IV., and known as *four figure logarithms*, is very convenient. By means of the numbers 10 to 99, and (*a*) those at the top of the table, and (*b*) in the difference column on the right, the logarithm of any number from 0 to 10000 can be written down.

In logarithms all numbers are expressed by the powers of some number called the *base*.

DEF.—**The logarithm of a number to a given base is the index showing the power to which that base must be raised to give the number.**

Thus if N denote any number and a the base, then by raising a to some power x we can get N. This is expressed by the equation $N=a^x$.

Any number could be used as the base, but, as we shall find, the system of logarithms in which the base is 10 is commonly used.

Thus, if the base be 2, then as $8 = 2^3$, 3 is the logarithm of 8 to the base 2, which can also be expressed by writing $\log_2 8 = 3$.

In a similar manner if the base be 5, then 3 is the logarithm of 125 to base 5; $\therefore \log_5 125 = 3$.

Also $64 = 2^6 = 4^3 = 8^2$.

Thus 6 is the log of 64 to the base 2;

also 3 is the log of 64 to the base 4;

and 2 is the log of 64 to the base 8, etc.;

$$\therefore \log_2 64 = 6;\ \log_4 64 = 3;\ \log_8 64 = 2, \text{ etc.}$$

The system of logarithms employed by the discoverer of logarithms, Napier, and called the **Napierian system,** is much used. The base of the Napierian system is denoted by the symbol e, and is the number which is the sum of the series

$$2 + \frac{1}{2} + \frac{1}{2 \times 3} + \frac{1}{2 \times 3 \times 4} + \dots.$$

This sum to 8 figures is 2·7182818.

Transformation of Logarithms.—A system of logarithms calculated to a base a may be transformed into another system in which the base is b.

Let N be a number, its logarithm in the first system we may denote by x, and in the second system by y.

Then $$N = a^x = b^y;$$

$$\therefore\ a^x = b^y; \qquad \therefore\ b = a^{\frac{x}{y}}.$$

Or $$\frac{x}{y} = \log_a b,$$

and $$\frac{y}{x} = \frac{1}{\log_a b}.$$

Hence, if the logarithm of any number in the system in which the base is a be multiplied by $\frac{1}{\log_a b}$, we obtain the logarithm of the number in the system in which the base is b.

The common logarithms or logarithms have been calculated from the Napierian logarithms. Thus let l and L be the logarithms of the same number in the common and Napierian systems respectively, then

$$l=\frac{1}{\log_e 10}L,$$

$$\log_e 10=2{\cdot}30258509,$$

and $$\frac{1}{2{\cdot}30258509}={\cdot}43429448.$$

Hence, the common logarithm of a number may be obtained by multiplying the Napierian logarithm of the same number by ·43429....

To convert common into Napierian logarithms we multiply by 2·3026 instead of the more accurate number 2·30258509.

Logarithms to the Base 10.—It is most convenient to use 10 as the base for a system of logarithms. It is only necessary to print in a table of such logarithms the decimal part or mantissa ; the characteristic can, we shall see, be determined by inspection. The tables are in this way less bulky than would otherwise be the case. When calculated to a base 10, logarithms are known as Common Logarithms.

Hence $N=10^x$; $\therefore\ \log_{10}N=x$,

or by definition, substituting positive numbers for N,

$1=10^0$; $\therefore$ log 1 = 0.

Also $10=10^1$; $\therefore$ log 10 = 1.

Again $100=10^2$; $\therefore$ log 100 = 2, etc.

In the chapter on Indices (p. 62) we have found that ·1, or $\frac{1}{10}$, is written as 10^{-1} ; also ·01, or $\frac{1}{100}$, is written as 10^{-2}.

Hence log ·1 = log $\frac{1}{10}=-1$,

and log ·01 = -2, etc.

The mantissa is always positive and instead of writing the negative sign in front of the number it is customary in logarithms to place it over the top, thus log ·1 is not written -1 but as $\bar{1}$, and log ·01 = $\bar{2}$.

In the preceding logarithms we have only inserted the characteristic, the mantissa consisting of a series of ciphers.

Thus log 1 = 0·0000,
log 10 = 1·0000,
log 100 = 2·0000, and so on.

As the log of 1 is zero, and log 10 is 1, it is evident that the logarithms of all numbers between 1 and 10 will consist of a certain number of decimals.

Thus log $2 = {\cdot}3010$ indicates that if we raise 10 to the power ·3010 we shall obtain 2, or $10^{\cdot 3010} = 2$. And in a similar manner the logarithm of $200 = 2 \times 100$ might be written as $10^2 \times 10^{\cdot 3010}$, $\therefore\ 200 = 10^{2 \cdot 3010}$.

Hence we write $\log 200 = 2{\cdot}3010$.

Rules of Logarithms.—(i) *The logarithms of two numbers added together gives the logarithm of the product of these numbers.*

Let a and b be the numbers.

Let $\log a = x$ and $\log b = y$;

$$\therefore\ a = 10^x,\ b = 10^y.$$

$$a \times b = 10^{x+y},$$

or $$\log_{10} ab = x + y = \log a + \log b.$$

(ii) *The logarithm of one number subtracted from the logarithm of another gives the logarithm of the quotient which is obtained by dividing the latter number by the former.*

As before let a and b be the two numbers.

Let $\log a = x$ and $\log b = y$;

$$\therefore\ a = 10^x,\ b = 10^y.$$

Hence $$\frac{a}{b} = \frac{10^x}{10^y} = 10^{x-y},$$

or $$\log_{10} \frac{a}{b} = x - y = \log a - \log b.$$

(iii) *The logarithm of the power of a number is the product of the logarithm of the number by the index of the power.*

Let $\log a = x$.

Then $a = 10^x$.

And $a^n = (10^x)^n$;

$$\therefore\ \log_{10} a^n = nx = n \log a.$$

The characteristic of a Logarithm.—It will be seen by reference to Table IV. that the logarithms of numbers are larger as the numbers are larger.

Thus, referring to Table IV., opposite the number 47 we find the *mantissa* ·6721, and as 47 lies between 10 and 100 the characteristic is 1. Hence the log of 47 is 1·6721.

Again, the number 470 lies between 100 and 1000, and therefore the characteristic is 2;

$$\therefore\ \log 470 = 2{\cdot}6721.$$

In a similar manner the logarithms of 4700 and 47000 are 3·6721 and 4·6721 respectively; in each case the mantissa is the same but the characteristic is different.

The rule by which the characteristic is found is as follows: *The characteristic of any number greater than unity is positive, and is less by one than the number of figures to the left of the decimal point.*

It must also be borne in mind that the *characteristic of a number less than unity is negative and is greater by one than the number of zeros which follow the decimal point.*

Ex. 1. log ·047. Here one zero follows the decimal point, hence the characteristic is $\bar{2}$;

$$\therefore\ \log {\cdot}047 = \bar{2}{\cdot}6721.$$

This may be verified by noting that

$$\log {\cdot}047 = \log \frac{4{\cdot}7}{100} = \log 4{\cdot}7 - \log 100$$

$$= {\cdot}6721 - 2 = \bar{2}{\cdot}6721.$$

Again, to obtain the log of ·00047.

As there are three zeros following the decimal point, the characteristic is $\bar{4}$, and $\therefore \log {\cdot}00047 = \bar{4}{\cdot}6721$.

Similarly log ·47. Here there are no zeros, and one greater will give $\bar{1}$ for the characteristic;

$$\therefore\ \log {\cdot}47 = \bar{1}{\cdot}6721.$$

This may also be verified by writing ${\cdot}47 = \frac{4{\cdot}7}{10}$;

$$\therefore\ \log \frac{4{\cdot}7}{10} = \log 4{\cdot}7 - \log 10$$

$$= {\cdot}6721 - 1 = \bar{1}{\cdot}6721.$$

Other numbers should be taken from Table IV., and the characteristic determined by inspection. After a little practice the writing down of the logarithm of any number becomes methodical and easy.

Another and better method of readily determining the characteristic is to write any given number as follows:

$$470 = 4{\cdot}7 \times 100 = 4{\cdot}7 \times 10^2.$$

Hence as before the characteristic is 2.

Similarly $4700 = 4{\cdot}7 \times 10^3$ $\quad {\cdot}47 = 4{\cdot}7 \times 10^{-1}$

$${\cdot}047 = 4{\cdot}7 \times 10^{-2} \quad {\cdot}0047 = 4{\cdot}7 \times 10^{-3}.$$

If all numbers are written in the above convenient form the characteristic is, as already indicated, the index of the multiplier 10.

If the method can be applied easily and readily by the student it will save the trouble of remembering rules, and the method is moreover extremely useful for many other purposes (see p. 63).

Logarithmic Tables.—To obtain the logarithm of a number consisting of four figures we proceed as follows:

Ex. 1. Find from Table IV. log 3768.

First look in the table for the number 37, then the next figure 6 is found at the top of table, so that the mantissa of

$$\log 376 = {\cdot}5752.$$

At the extreme right of the table will be seen a column of differences, as they are called; thus under the figure 8 is found the number 9. This must be added to the mantissa previously obtained.

Hence we have mantissa of log 376 = 5752

Add difference, $\underline{9}$

Gives mantissa for log 3768 = 5761

Hence $\log 3768 = 3{\cdot}5761,$

also $\log {\cdot}003768 = \bar{3}{\cdot}5761,$

and $\log {\cdot}3768 = \bar{1}{\cdot}5761,$ etc.

To find the number corresponding to a given logarithm or the antilogarithm of a number.

Ex. 2. Given the logarithm 2·4725 find the number.

From Table IV., of Antilogarithms.

Opposite the mantissa ·472 we have 2965. In the difference column under the number 5 and on the horizontal line 47 we have the figure 3.

Hence the corresponding mantissa = 2968, and the number required is 296·8.

If the given logarithm had been $\bar{2}{\cdot}4725$ the required number would be ·02968.

Multiplication by Logarithms.—*By adding the logarithms of two or more numbers together, we obtain the logarithm of their product and the number corresponding to this logarithm obtained from Table V. of antilogarithms is the product required.*

Ex. 1. Multiply 598·6 by ·0546.

From Table IV. mantissa corresponding to 598 = 7767.

Diff. col. for 6, 4.

log 598·6 = 2·7771.

Mantissa corresponding to 546 = ·7372;

$\therefore$ log ·0546 = $\bar{2}$·7372;

$\therefore$ logarithm of product = $\bar{2}$·7771 + 2·7372

= 1·5143.

Antilogarithm corresponding to log 514 = 3266

Diff. col. for 3 = 2

3268

Hence number corresponding to 1·5143 = 32·68;

$\therefore$ 598·6 × ·0546 = 32·68.

EXERCISES. XLII.

Using four-figure logarithms, multiply together and find in each case the product of the following:

1. 20·30 by ·0635.

2. 8·803 by ·065 and by ·00842.

3. 5·39, 21·07, and ·013.

4. 4·06, 72·1, and ·0013.

5. 3·17 by 4·02, and their product by ·0341.

6. 12·607 by 3·08, and their product by 1·0006.

7. 67·891 by ·04508.

Find the continued product of the following:

8. 15·08, 6·47, and ·0905.

9. 26·55, 3·067, and ·0938.

10. 33·77, 401·3, and ·00987.

11. 30·75, 29·417, and ·00123.

12. 63·05, 1·2037, and ·00089.

13. 45·61, 4·0045, and ·00017.

14. 6·325, 40·83, and ·00253.

Division by logarithms.—*The logarithm of the quotient is obtained by subtracting the logarithm of the divisor from the logarithm of the dividend; the number corresponding to this logarithm found on reference to the table of antilogarithms is the number required.*

Ex. 1. Divide 30·48 by ·0525.

Mantissa of log 304 = 4829

Diff. col. for 8 = 11

Mantissa of log 3048 = 4840

$\therefore$ log 30·48 = 1·4840

In a similar manner we obtain log ·0525 = $\bar{2}$·7202

Subtracting 2·7638

Referring to table of antilogarithms—

Corresponding to 763 we have 5794
Diff. col. for 8, 11
5805

Hence 580·5 = log 2·7638;
∴ 30·48 ÷ ·0525 = 508·5.

EXERCISES. XLIII.

Divide

1. ·9651 by 187·5.
2. 12·499 by 414·3.
3. ·4990 by ·00608.
4. 91·77 by ·0399.
5. 1·6308 by 362·4, and multiply the result by 1·942.
6. 124·5 by ·0083.
7. ·08936 by 558·5.
8. 98·5 by ·0000394.
9. 64·01 by ·0346.
10. Find the value of $\frac{12{\cdot}08}{{\cdot}01299} \times {\cdot}02112$.

Divide

11. 33·51 by ·0713.
12. ·03081 by 237.
13. 298·08 by ·00345.
14. 1 by ·00375.
15. ·036387 by 1·17.
16. 157·5 by ·03706.
17. Multiply ·0047 by ·00035, and divide ·00918 by ·018.
18. Divide ·0053 by 2·5, and 53 by ·0025.
19. State and prove the rule for finding the logarithm of the quotient when the logarithms of the dividend and the divisor are given.
20. Define the characteristic of a logarithm, and state the rule by which the characteristic of the logarithm to the base 10 of a given number may be written down by inspection. Write down the logarithms of 5374, 5374000, and 0·005374.
21. (i) Define the logarithm of a number, and show from the definition that $\log\frac{ab}{c} = \log a + \log b - \log c$.

    (ii) Find the logarithms of $7\frac{7}{9}$ and 0·0105.
22. (i) Add together $\bar{8}{\cdot}2153$, $6{\cdot}3213$, and $\bar{3}{\cdot}3726$.

    (ii) Divide $\bar{8}{\cdot}5503$ by 7.

23. (i) Subtract $\bar{3}\cdot1528$ from $\bar{2}\cdot4985$. (ii) Divide $\bar{2}\cdot5188$ by 6.

24. Multiply ·02019 by 52·03, and divide ·04312 by ·0044.

25. Multiply and divide 2014 by 3·128.

26. Find the logarithm of $97\cdot94 \times 0\cdot006386$.

27. Write down log 106·1 and log 0·001061.

Involution by Logarithms.—*To obtain the power of a number, multiply the logarithm of the number by the index of the power required; the product is the logarithm of the number required.*

Ex. 1. Calculate the value of $(0\cdot07)^3$.

$$\begin{aligned} \log \cdot07 &= \bar{2}\cdot8451 \\ &\phantom{=\bar{2}\cdot845}3 \\ &\overline{\bar{4}\cdot5353} \end{aligned}$$

The process is as follows: Writing down the log of the number as shown, we multiply by the index (3) of the power, and obtain for the mantissa ·5353 and the characteristic $\bar{4}$. This always presents some difficulty to a beginner. To obtain the characteristic we say $3 \times 8 = 24$, plus 1 carried from last figure, gives 25, and we write down 5; next, $3 \times (-2) = -6$, and -6 added to $+2$ carried from previous figure gives -4, which, as already described, is written $\bar{4}$.

$$\begin{aligned} \therefore \quad \log(0\cdot07)^3 &= 3 \times \bar{2}\cdot8451 \\ &= \bar{4}\cdot5353. \end{aligned}$$

Referring to table of antilogarithms—

Corresponding to 535 we find	3428
Diff. col. for 3 we find	2
	3430

Hence $\bar{4}\cdot5353 = \cdot000343$.

Evolution by Logarithms.—*The logarithm of the number whose root is required is divided by the number which indicates the root.*

It should be noted that, although the index of the logarithm may be negative, the mantissa remains positive. Hence the logarithm when negative usually requires a little alteration in form before dividing by the number so as to be exactly divisible by it.

The method adopted can best be shown by an example.

Ex. 1. Find the 4th root of 0·007 or $(\cdot 007)^{\frac{1}{4}}$.

$$\log \cdot 007 = \bar{3}\cdot 8451.$$

$$\tfrac{1}{4}(\bar{3}\cdot 8451) = \tfrac{1}{4}(\bar{4} + 1\cdot 8451)$$

$$= \bar{1}\cdot 46127 = \bar{1}\cdot 4613.$$

$$\begin{array}{lr} \text{Corresponding to the mantissa 461 we find the antilogarithm} = & 2891 \\ \text{Diff. col. for } 3 = & \underline{\quad 2} \\ & 2893 \end{array}$$

∴ the antilogarithm corresponding to the logarithm $\bar{1}\cdot 4613$ is ·2893.

$$\text{Hence } (0\cdot 007)^{\frac{1}{4}} = \cdot 2893.$$

Ex. 2. Find the 7th root and the 7th power of 0·9306.

$$\log \cdot 9306 = \bar{1}\cdot 9688,$$

$$\text{log of 7th root} = \tfrac{1}{7}(\bar{7} + 6\cdot 9688) = \bar{1}\cdot 9955.$$

Referring to table of antilogarithms, we find

$$\begin{array}{lr} \text{Antilogarithm of } 995 = & 9886 \\ \text{Diff. col. for 5,} & \underline{\quad 11} \\ \therefore \text{ antilogarithm of } \cdot 9955 = & 9897 \end{array}$$

The characteristic $\bar{1}$ indicates that the number is less than unity.
Hence 7th root = ·9897.

Let x denote the 7th power of ·9306

Then $x = (0\cdot 9306)^7$.

$$\therefore \log x = 7 \log \cdot 9306$$

$$= 7 \times \bar{1}\cdot 9688 = \bar{1}\cdot 7816.$$

$$\begin{array}{lr} \text{Antilog } 781 = & 6039 \\ \text{Diff. col. for 6,} & \underline{\quad 8} \\ & 6047 \end{array} \quad \text{Hence } x = \cdot 6047.$$

Having in the preceding pages indicated how, by means of logarithms, many tedious processes are materially shortened, and how results in these and also in operations impossible by arithmetical methods can be obtained with little trouble, it now remains to show the principles on which tables of common and hyperbolic logarithms may be constructed. Usually this is taken to mean an advanced knowledge of Algebra and algebraic processes, including "series" and the exponential theorem. Prof. Perry has shown, in a letter sent to *Nature*, February 23, 1899, how such a table may readily be constructed, by any one familiar with the mode of finding the square root in Arithmetic. The method was devised independently by Mr. E. Edser, whose claim to priority was acknowledged in a subsequent letter to *Nature.* The plan adopted is very simple and direct, as may be seen by the following abstract of Prof. Perry's description.

To calculate a table of logarithms.—Let the student find $10^{\frac{1}{2}}$, $10^{\frac{1}{4}}$, $10^{\frac{1}{8}}$ $10^{\frac{1}{16}}$, $10^{\frac{1}{32}}$ by repeatedly extracting square roots, getting out the answers to five significant figures, say. Let him now by multiplication calculate $10^{\frac{3}{32}}$, $10^{\frac{5}{32}}$, etc., right up to 10^{1}. He thus has a table of which the beginning and end are here given.

Number.	Logarithm.
1 0000	0·00000
1 0746	0·03125
1·1548	0·06250
1·2409	0·09375
1·3336	0·12500
...	...
...	...
...	...
8·0584	0·90625
8·6596	0·09750
9·3057	0·96875
10·0000	1·00000

If these values are plotted on squared paper a curve is obtained which enables the logarithm of any number to be determined.

If the answers are required right to four significant figures, it is well to draw only a small part of the curve on one sheet of squared paper. Thus plotting the first three points so that the curve joining them (using a slightly bent straight-edge) passes diagonally through a small sheet of squared paper, answers may be obtained which are sufficiently correct to impress a student with the value of the method. Seven values taken at random from the curves of the early and the late parts of a table thus constructed by a student are here given This was the first time that he had tried the method, and the small errors in the fourth significant figures are not likely to occur after a little practice.

Number.	Logarithm as measured.	Correct logarithm.
1·035	·0149	·0149
1·110	·0452	·0453
1·151	·0610	·0611
8·950	·9520	·9518
9·175	·9626	·9626
9·345	9706	·9705
9·825	·9923	·9923

Even with the very cheapest squared paper tables of logarithms and antilogarithms can be constructed which will be quite accurate to three significant figures, and by taking more trouble and using $10^{\frac{1}{64}}$ a table accurate to four significant figures can be obtained even with very cheap paper. Also it is to be noticed that by using $10^{\frac{1}{128}}$ and higher roots, the logarithm of any particular number can be determined with any amount of accuracy desired.

EXERCISES. XLIV.

Find from the Table the logarithms of

1. $\sqrt{(0\cdot02)}$.
2. $(0\cdot02)^7$.
3. 1061·1.
4. ·00161.
5. $(0\cdot000596)^{\frac{1}{3}}$.
6. $(\cdot00112)^{\frac{1}{4}}$.
7. 72.
8. 14·4.
9. ·00015.
10. $\sqrt[3]{\left(\frac{3^2 5^4}{\sqrt{2}}\right)}$.
11. $\frac{15}{\sqrt{91}}$; $\frac{1\cdot025}{(0\cdot027)^2}$; $\sqrt{0\cdot0013} : \sqrt{\cdot0002178}$.
12. $\sqrt[3]{84}$; $\sqrt{0\cdot0125}$; $\frac{(1\cdot08)^4}{(0\cdot0147)^3}$.
13. 3000, $\frac{1}{3000}$, and $2000 : \frac{2}{10000}$.
14. ·0128 and $(\cdot064)^{\frac{1}{4}}$.
15. $\frac{4\cdot5}{0\cdot000018}$.

16. Find the 7th root and the 7th power of 0·9306 to five decimal places.

17. Find the logarithm of 10 to base 2.
What is the logarithm of $\sqrt{2}$ to base 2?
Find logarithm of 64 to base 16.

18. Calculate the value of $(0{\cdot}007)^{\frac{1}{4}} \div (0{\cdot}07)^3$.

19. Find the 7th root of $\frac{13}{300}$.

Calculate by means of logarithms the numerical values of the following:

20. $\left(\frac{347}{123}\right)^{\frac{1}{5}}$. **21.** $\sqrt[5]{{\cdot}023476}$. **22.** $\left(\frac{5}{7}\right)^{{\cdot}1345}$.

23. $\sqrt[5]{{\cdot}0103}$. **24.** $\frac{(7432)^{\frac{3}{7}}}{\sqrt{29}}$.

25. (i) $\left\{\frac{5\sqrt[3]{138}}{\sqrt[5]{{\cdot}01}}\right\}^{\frac{1}{2}}$. (ii) $a^{\frac{1}{5}} \times b^{\frac{1}{3}}$, when $a=34$, $b=24{\cdot}68$. **26.** $\frac{\sqrt[3]{172500}}{\sqrt[5]{{\cdot}01}}$.

27. The fifth root of 14·52 and the cube root of ·000048.

28. $\frac{a^{\frac{1}{3}} \times b^{\frac{3}{4}}}{c^{\frac{1}{2}}}$, when $a=15{\cdot}6$, $b={\cdot}0045$, $c={\cdot}00065$

29. $\sqrt{2543 \times {\cdot}1726}$.

30. $\frac{a^{\frac{1}{3}} \times b^{\frac{1}{2}}}{c}$, when $a={\cdot}0125$, $b=31{\cdot}15$, $c={\cdot}00081$.

31. The square root of $\frac{17}{32}$. **32.** $\frac{(1{\cdot}086)^{11}-(1{\cdot}024)^{11}}{(365{\cdot}73)^{\frac{1}{7}}}$.

33. $\sqrt[3]{{\cdot}038 \times (1{\cdot}03847)^4}$. **34.** $\frac{(7{\cdot}25)^{\frac{1}{5}} \times 1{\cdot}0046}{({\cdot}0874)^2}$.

35. Find a fourth proportional to 254·5, 4000·6, and ·028.

36. $\frac{1}{(1{\cdot}44)^3}$.

37. Find (i) $\log \sqrt[5]{{\cdot}00054 \times 3{\cdot}6}$, (ii) $\log \frac{1}{324\sqrt[5]{125}}$.

(iii) How many cyphers are there between the decimal point and the first significant digit in $({\cdot}0504)^{10}$ expressed as a decimal?

38. Calculate $a \times b$ and $a \div b$,
(i) when $a=2646$, $b=48{\cdot}64$,
(ii) ,, $a=35{\cdot}12$, $b=287$,
(iii) ,, $a=1{\cdot}128$, $b=4942$.

39. Find the values of $\cdot03571 \times \cdot2568$ and $8352 \times 3{\cdot}69 \div (30{\cdot}57)^3$.

40. Find the logarithms of $\sqrt{2}$, 4, 16, 64 to base 16.

MISCELLANEOUS EXERCISES. XLV.

Find the numerical values of the following:

1. $\sqrt[5]{(0\cdot02)}$ and $\dfrac{1}{(1\cdot5866)^3}$.

2. $\dfrac{3\cdot745 \times \cdot0134}{987\cdot7}$ and $\sqrt{\dfrac{\cdot9795 \times \cdot0125}{6\cdot36}}$.

3. $31\cdot09^3$ and $\cdot000526^{\frac{1}{3}}$, also $32\cdot83 \times (1\cdot015)^{12}$.

4. $(0\cdot007)^{\frac{1}{4}} \div (0\cdot07)^3$.

5. Find the numerical value of the fifth root $\dfrac{\sqrt{2}}{3^3}$.

6. Find the fifth root of $(0\cdot0002 \div 23087)$.

7. Find the numerical value of $(18)^{\sqrt{2}}$.

8. How many ciphers are there between the decimal point and the first significant figure in $(\frac{1}{3})^{100}$?

9. Find the fifth root of $0\cdot0102$.

10. Show that $(\frac{21}{20})^{50}$ is greater than 10.

11. Find the value of $\sqrt[7]{\cdot00792}$.

12. Determine which is greater, $0\cdot1$ or $(\frac{4}{5})^{10}$.

13. Find the logarithm (with a positive mantissa) of the cube root of $0\cdot0625$, and calculate the value of $(0\cdot03)^5 \div (0\cdot12)^4$.

14. Find the logarithms of $(\frac{1}{200})^{11}$, and of $(0\cdot000216 \times 5\cdot4)^{\frac{1}{7}}$.

15. Find the logarithms of 25, $\cdot03125$, and $(\cdot025)^{\frac{1}{5}}$.

16. $\left(\dfrac{\sqrt{2}}{27}\right)^{\frac{1}{5}}$.

17. Find the logarithms of $1\cdot4$, $\cdot0154$, and $\dfrac{2^{10}}{7^9 \times 11^5}$.

18. Find x, having given that $\frac{2}{3}$ is the logarithm of x to the base 8.

Find the common logarithm (with a positive mantissa) of the square root of $0\cdot00119$, and calculate the numerical value of $(0\cdot084)^{\frac{1}{5}} \div (0\cdot34)^3$.

SUMMARY.

The **logarithm** of *a number* (abbreviated to log) *to a given base is the index of the power to which that base must be raised to give the number.* Thus, if the base be 10, the log of $100 = 2$, since $10^2 = 100$.

Two systems.—There are two systems of logs in general use; these are known as *common* in which the base is 10, and *Napierian*, or (as they are often called) *hyperbolic*, in which the base is 2·71828.... This base is usually denoted by the letter *e*.

Mantissa and Characteristic.—The log of a number consists of two parts called the mantissa and the characteristic; the former may be obtained from Table IV., the latter is determined by inspection.

Multiplication.—Add together the logs of the numbers, and find the number whose antilog is their sum.

Division.—Subtract the log of the divisor from the log of the dividend and find the number whose antilog is the difference.

Involution.—To raise a number to a given power, multiply the log of the number by the index of the power, and find the number the antilog of which is the product.

Evolution.—Divide the log of the number by the index of the root required (first making, if necessary, the requisite adjustment in the characteristic). The quotient is the antilog of the root.

CHAPTER XI.

MEASUREMENT AND FUNCTIONS OF ANGLES. SINE, COSINE AND TANGENT.

WE have already referred to the use of a protractor in measuring the inclination of one line which meets another at a given point.

One of the most useful methods of forming an estimate of the magnitude of angles is by means of the so-called functions of angles, the sine, cosine, tangent, etc. The values of these functions have been tabulated for every degree and every minute up to 90°. It is possible, in addition, by means of columns of differences to calculate these functions to any number of seconds.

It will obviously be an advantage to clearly understand what information is directly obtainable from such tables, and how to use them with accuracy and facility.

If at any point B (Fig. 54) along the line AD a line BC be drawn perpendicular to AD and meeting AE in C, then ABC is a right-angled triangle; one angle at B, a right angle, is known in magnitude; the other two angles and the three sides are determined by the data of any given question.

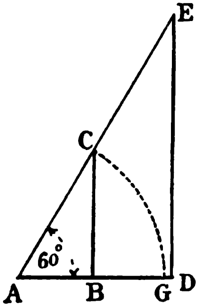


FIG. 54.—Angle of 60°.

If one of the sides be given, and one of the two remaining angles, the other sides and the remaining angle can be found either by construction, or by calculation.

Thus, given the angle BAC and one of the sides AB, BC or AC, as will be seen on reference to Fig. 54, the triangle can be constructed.

Also any two of the three sides will give sufficient data to enable the triangle to be drawn and the two angles and the remaining side to be found.

It is not necessary to give the actual lengths of two of the sides; it will answer the same purpose if the ratio of AB to BC be known, for if any line such as DE be drawn parallel to BC it will cut off lengths AD and DE from the two sides AB and AC produced such that the ratio of AD to DE is equal to the ratio of AB to BC.

Also the equality is unaltered if BC and DE be replaced by AC and AE respectively.

This statement can be written $\frac{AB}{AC}=\frac{AD}{AE}$.

To obtain the value of the ratio $\frac{AB}{AC}$ *when the angle is* 60°.—Draw any line AD as base (Fig. 54), and with A as centre and a radius of 10 units describe an arc GC. With G as centre and same radius draw an arc intersecting the former at C; join A to C; then DAC is an angle of 60°.

From C draw CB perpendicular to AD and meeting AD at B; ascertain as accurately as possible the lengths of AB and BC.

AB will be found to be 5 units, and BC to be 8·66 units.

Hence the ratio $\frac{AB}{AC}=\frac{5}{10}=\frac{1}{2}$, and $\frac{BC}{AC}=\frac{8{\cdot}66}{10}$.

This ratio of $\frac{BC}{AC}$ is called the **sine** of the angle BAC or (as only one angle is formed at A) sine A.

It will be seen from the above that the *sine* of an angle (which is abbreviated into *sin.*) is formed by the *ratio of two sides of a right angled triangle; the side opposite the angle being the numerator, and the hypothenuse or longest side of the triangle adjacent to the angle the denominator.*

We can hence write $\sin 60^\circ=\frac{\text{side } BC}{\text{side } AC}=\frac{8{\cdot}66}{10}={\cdot}866$.

The length of the side BC can also be obtained by calculation, *i.e.* in the right-angled triangle ABC we have

$$BC^2=AC^2-AB^2=10^2-5^2=75;$$

$$\therefore\ BC=\sqrt{75}=8{\cdot}66.$$

The ratio of $\frac{AB}{AC}$ is called the **cosine** of the angle BAC. (Cosine is abbreviated into *cos.*)

Hence $\cos 60^\circ = \frac{5}{10} = \frac{1}{2}$.

We have already found (p. 34) that the sum of the three angles of any triangle is 180°. As one of the angles in Fig. 54 is 90° and the other 60°, the remaining angle ACB is 30°. Whenever we know the magnitude of each side of the triangle ACB, the values of the sine, cosine and tangent of 30° can be found; or we may proceed as follows:

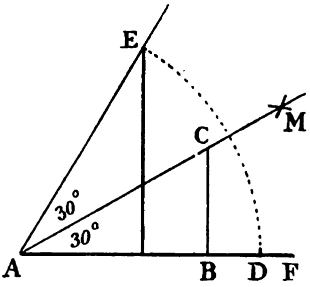


FIG. 55.—Angle of 30°.

To construct an angle of 30°: Draw a line AF (Fig. 55), and with A as centre and any convenient radius AD describe an arc of a circle DE; with the same radius and D as centre draw an arc intersecting DE at E; by joining A to E an angle of 60° is formed. Now bisect this angle of 60°. with D and E as centres and any convenient radius describe arcs intersecting in M. Join M to A. Then $FAM = MAE = 30^\circ$.

Make $AC = 10$ units on any scale. Drop a perpendicular from C cutting AF at B. Measure and write down the lengths AB and BC to the same scale.

The Tangent.—Of the three sides of the triangle AB, BC, and CA we have already taken the ratio of $\frac{BC}{AC}$ and $\frac{AB}{AC}$, the former called the sine and the latter the cosine of θ (Fig. 56). One other ratio, and a most important one, is the ratio of $\frac{BC}{AB}$. This ratio is called the tangent of BAC, or denoting BAC by θ, and using the abbreviation tan for tangent, we have $\tan \theta = \frac{BC}{AB}$.

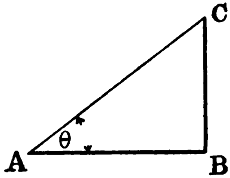


FIG. 56.

Ex. 1. Set out an angle of 60°. Make AC (Fig. 54) to any convenient scale equal to 10 units, then BC will be found to be 8·66 units;

$$\therefore \quad \tan 60^\circ = \frac{8{\cdot}66}{5} = 1{\cdot}732. \quad \text{Also } \tan 30^\circ = \frac{AB}{BC} = \frac{5}{8{\cdot}66} = {\cdot}577.$$

Verify these results by referring to Table VI., p. 376.

Also as $$\sin\theta = \frac{BC}{AC}, \text{ and } \cos\theta = \frac{AB}{AC},$$

$$\therefore \quad \frac{\sin\theta}{\cos\theta} = \frac{\frac{BC}{AC}}{\frac{AB}{AC}} = \frac{BC}{AB} = \tan\theta,$$

or, the numerical value of the sine of any angle divided by that of the cosine of the same angle is equal to the numerical value of the tangent of the angle.

Angle of 45°.—Draw a line AB, and using the set square, draw a line EF at 45° with EB, cutting BF at F (Fig. 57). It will be found that BE is equal to BF. Make $EG = 10$ units, and draw GC parallel to EB, cutting BF at C. Draw CA parallel to FE; then the triangle BAC is a right-angled isosceles triangle, each of its equal angles at A and C is 45°, and the hypothenuse AC is 10 units in length.

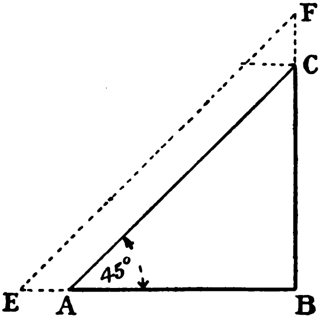


Fig. 57.—Angle of 45°.

As AB and BC are equal, it follows that $\tan 45^\circ = \frac{AB}{BC} = 1$.

Ex. 1. Construct angles of 30°, 45°, and 60°. In each case make the hypothenuse $AC=10$ units on any convenient scale. Measure the lengths AB and BC to the same scale, and tabulate as follows:

	Lengths of:		Numerical values of:			sin.	cos.	tan.
	AB.	BC.	$\frac{BC}{AC}$.	$\frac{AB}{AC}$.	$\frac{BC}{AB}$.			
Angle of 30°,								
Angle of 45°,								
Angle of 60°,								

The **complement** of an acute angle is the angle by which it falls short of a right angle. Thus, the complement of an angle of 60° is 30°. In a similar manner the complement of 30° is 60°.

Hence, two angles are said to be **complementary** when their sum is a right angle.

Referring to Fig. 55, the sin of $30° = \cos 60°$, the ratio in each case being $\frac{BC}{AC}$. Hence *the sine of an angle is the cosine of the complement of that angle*; *and the cosine of an angle is the sine of the complement of that angle.* Prove these by reference to the tabulated values obtained by measurement.

The **supplement** of an angle is the angle by which it falls short of two right angles, or 180°; thus, the supplement of an angle of 60° is 120°; the supplement of an angle of 30° is 150°.

Hence two angles are said to be **supplementary** when the sum of the two angles is 180°.

It is important to be able to readily write down the values of the sine, cosine, etc., of angles of 60° and 30°.

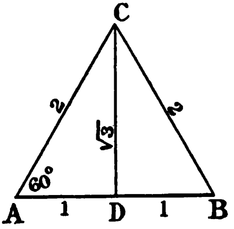


FIG. 58.—Equilateral triangle.

To do this, the best method is to draw, or *mentally picture*, an equilateral triangle ABC (Fig. 58), each side of which is 2 units in length.

From the vertex C let fall a perpendicular CD on the base AB. As shown on p. 35, the point D bisects AB, and $AD=DB$. Also the angle ACD is equal to DCB; each of these equal angles is one-half the angle ACB, and is therefore 30°; or, as the angle at D is 90°, and the angle at C is equal to 60°, the remaining angle ACD must be 30°.

From the right-angled triangle ADC, we have

$$DC^2=AC^2-AD^2=4-1=3;$$

$$\therefore\ DC=\sqrt{3}.$$

Thus, $$\sin 60^\circ=\frac{DC}{AC}=\frac{\sqrt{3}}{2};\quad \cos 60^\circ=\frac{AD}{AC}=\frac{1}{2};$$

$$\tan 60^\circ=\frac{DC}{AD}=\frac{\sqrt{3}}{1}=\sqrt{3}=\frac{\sin 60^\circ}{\cos 60^\circ}$$

Hence, $$\sin 30^\circ=\frac{AD}{AC}=\frac{1}{2}=\cos 60^\circ;$$

$$\cos 30^\circ=\frac{DC}{AC}=\frac{\sqrt{3}}{2}=\sin 60^\circ;$$

$$\tan 30^\circ=\frac{AD}{DC}=\frac{1}{\sqrt{3}}=\cot 60^\circ.$$

It should be noticed that

$$\frac{\sin 60^\circ}{\cos 60^\circ}=\frac{DC}{AC}\div\frac{AD}{AC}=\frac{DC}{AD}=\tan 60^\circ.$$

In a similar manner, for any angle A,

$$\frac{\sin A}{\cos A}=\tan A.$$

Instead of attempting to remember the important numerical values for the ratios, it will be found much better to use the triangle, as described in Fig. 58, its angles 90°, 60°, and 30°, and its sides in the proportion of 2, 1, and $\sqrt{3}$.

To ascertain the numerical values of the sine, cosine, and tangent of 45°, a similar method may be used. Thus, if AB and BC (Fig. 59) are two lines at right angles, and BA is equal to BC, join A to C.

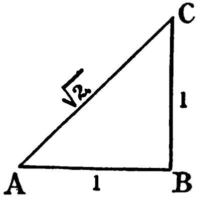


Fig. 59.—Isosceles right-angled triangle.

The side BA is equal to the side BC. Therefore, as shown on p. 35, the angle $BAC=BCA$, and each is 45°, the angle at B being 90°.

$$\text{Length of } AC=\sqrt{AB^2+BC^2}=\sqrt{2}.$$

Hence the three sides of the triangle ABC are in the ratio of 1, 1, and $\sqrt{2}$.

$$\text{Thus} \qquad \sin 45^\circ=\frac{BC}{AC}=\frac{1}{\sqrt{2}}; \quad \cos 45^\circ=\frac{AB}{AC}=\frac{1}{\sqrt{2}},$$

$$\text{or} \qquad \sin 45^\circ=\cos 45^\circ; \quad \tan 45^\circ=\frac{BC}{AB}=\frac{1}{1}=1.$$

Angles greater than 90°.—It is often necessary to consider angles greater than 90°; and in particular the ratios of angles from 0° to 180° are very important. Thus, if BAC (Fig. 60) be a given angle, and if with centre A and any convenient radius (as AC) a circle be described, then it will at once be seen by making

$$B'C'=BC,$$

and joining C' to A, that the triangle $B'AC'$ is exactly similar to BAC;

$$\therefore \sin B'AC'=\sin BAC.$$

Angles are always measured in an anti-clockwise manner, *i.e.* in the direction of the arrow, hence if BAC be an angle of 30°, as BAC' is equal to $180^\circ-30^\circ$,

$$\therefore \sin(180^\circ-30^\circ)=\sin 30^\circ, \text{ or } \sin 150^\circ=\sin 30^\circ.$$

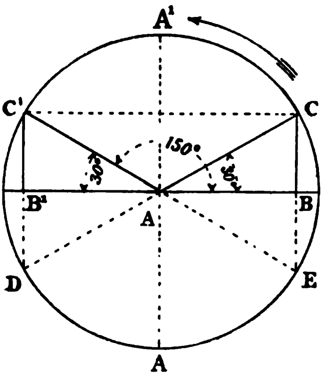


FIG. 60.—Angles greater than 90°.

In general, $\sin(180^\circ-A)=\sin A$.

In addition to the convention that all angles are measured in an anti-clockwise manner, the following rules are adopted:

All lines measured in an upward direction from BB' are positive, and all lines measured from $A'A$ towards B are positive; those in the opposite directions, *i.e.* downwards, or from $A'A$ to B', are negative. The radius, or side AC, or AC', is always positive.

Hence if BAC denote any angle A, as BC and $B'C'$ are both in an upward direction, we have, as before

$$\sin(180^\circ - A) = \sin A.$$

But AB and AB' are lines drawn in opposite directions;

$$\therefore \cos(180^\circ - A) = -\cos A.$$

In the case of the angle formed by producing $C'B'$ to D, both the sine and cosine of the angle are negative; when the angle is formed by producing CB to E, the sine of the angle is negative, the cosine is positive.

As the tangent is $\frac{\text{sine}}{\text{cosine}}$, its sign, positive or negative, will depend upon the signs of the sine and cosine of the angle; when these are alike the tangent is positive, and negative when they are unlike. Further, when the numerator is zero, the value of the tangent is 0; the value is indefinitely great when the denominator is 0; this is written as ∞.

The following important relations should be proved by drawing, as already described, a right-angled triangle to scale:

$$\sin 60^\circ = \frac{\sqrt{3}}{2} = \sin 120^\circ;$$

$$\cos 60^\circ = \frac{1}{2}, \quad \cos 120^\circ = -\frac{1}{2}.$$

$$\tan 60^\circ = \sqrt{3}, \quad \tan 120^\circ = -\sqrt{3}.$$

These results and the preceding, are collected in the following table, and should be verified.

	0°	30°	45°	60°	90°	120°	135°	150°	180°
sin	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{\sqrt{3}}{2}$	-1
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞	$-\sqrt{3}$	-1	$-\frac{1}{\sqrt{3}}$	0

The three ratios—sine, cosine, and tangent—are of the utmost importance, and should be written and re-written until they become quite familiar; from these, the other three ratios, cosecant, secant, and cotangent of an angle are easily obtained.

Thus, in any right-angled triangle, ABC (Fig. 61) denoting the angle BAC in the usual manner (by A).

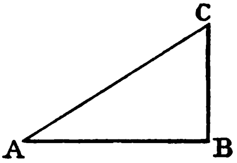


FIG. 61.

Then,

$$\textbf{cosecant A}=\frac{1}{\sin A}=\frac{1}{\frac{BC}{AC}}=\frac{AC}{BC};$$

$$\textbf{secant A}=\frac{1}{\cos A}=\frac{1}{\frac{AB}{AC}}=\frac{AC}{AB};$$

$$\textbf{cotangent A}=\frac{1}{\tan A}=\frac{1}{\frac{BC}{AB}}=\frac{AB}{BC}.$$

The above are usually designated as **cosec A, sec A,** and **cot A** respectively.

We can also prove the following important relations :

(i) $\mathbf{\sin^2 A+\cos^2 A=1}$.

For $$\sin A=\frac{BC}{AC};\quad \cos A=\frac{AB}{AC}.$$

Then $$(\sin A)^2+(\cos A)^2=\frac{BC^2}{AC^2}+\frac{AB^2}{AC^2}=\frac{BC^2+AB^2}{AC^2}=1;$$

or $$(\sin A)^2+(\cos A)^2=1.$$

Usually, $(\sin A)^2$ is written $\sin^2 A$.

And in a similar manner $(\cos A)^2=\cos^2 A$.

(ii) $\mathbf{\sec^2 A=1+\tan^2 A}$.

$$\sec A=\frac{AC}{AB};$$

$$\sec^2 A=\frac{AC^2}{AB^2}=\frac{AB^2+BC^2}{AB^2}=1+\frac{BC^2}{AB^2}=1+\tan^2 A.$$

Also $\mathbf{\operatorname{cosec}^2 A=1+\cot^2 A}$.

For $$\operatorname{cosec}^2 A=\frac{AC^2}{BC^2}=\frac{BC^2+AB^2}{BC^2}=1+\frac{AB^2}{BC^2}=1+\cot^2 A.$$

Values of sine, cosine, and tangent for angles of $22\frac{1}{2}°$ *and* $67\frac{1}{2}°$.

We can obtain angles of $22\frac{1}{2}°$ and $67\frac{1}{2}°$ by construction as follows :

At any point B on a straight line AB draw BC perpendicular to AB. Make $BA = BC$ (Fig. 62), join AC; as shown on p. 35 the angle $BAC =$ angle BCA, and each is 45°. Produce BA to D, making $AD = AC$; then as before, the two angles ADC and ACD are equal.

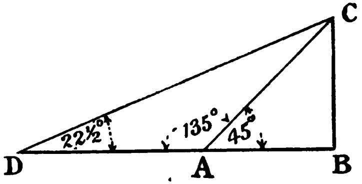


FIG. 62.—Angle of 22½° and 67½°.

The two angles at A are together equal to 180°, and as one of them, BAC is 45°, the remaining one, DAC, is 135°. In the triangle DAC the angles ADC and ACD are together equal to $180° - 135° = 45°$, and as the angles are equal, each is 22½°.

The angle $BCD = BCA + ACD = 45° + 22\frac{1}{2}° = 67\frac{1}{2}°$;

$$\sin 22\tfrac{1}{2}° = \frac{BC}{CD}; \quad \cos 22\tfrac{1}{2}° = \frac{DB}{CD}.$$

Also $$\sin 67\tfrac{1}{2}° = \frac{DB}{CD}; \quad \cos 67\tfrac{1}{2}° = \frac{CB}{CD}.$$

To obtain values of sine, cosine, and tangent for angles of 15° *and* 75°.

Draw any line as DB, and BC perpendicular to DB (Fig. 63). Make the angle $BAC = 30°$. As the angle at B is 90°, the angle $ACB = 60°$.

Produce BA to D, making $AD = AC$. Join DC. As shown on p. 35, as the side $AD =$ side AC, the angle $ADC = ACD$. Also as the straight line meets BD at A, the two angles BAC and DAC are together equal to 180°, and as $BAC = 30°$, the angle

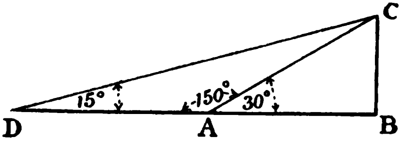


FIG. 63.—Angles of 15° and 75°.

$$DAC = 180° - 30° = 150°.$$

Again, the three angles of the triangle DAC are together equal to 180°, and as one of the angles is 150°, and the two remaining angles are equal, each is therefore equal to 15°.

Hence $$\sin 15° = \frac{BC}{DC}; \quad \cos 15° = \frac{DB}{DC}; \quad \tan 15° = \frac{BC}{DB}.$$

Referring to Fig. 63, the angle $DCB = 60° + 15° = 75°$;

$$\therefore \ \sin 75° = \frac{BD}{CD}; \quad \cos 75° = \frac{BC}{CD}; \quad \tan 75° = \frac{BD}{BC}.$$

Small angles.—*Ex.* When the size of an angle is small, show by drawing to a comparatively large scale, that the sine, tangent, and radian measure of the angle are very nearly equal.

EXERCISES. XLVI.

1. Draw an angle of 40°, making the hypothenuse 10 or 100 units. Find by measurement the values of the sine, cosine, and tangent of the angle; compare the values obtained with those in Table VI., p. 376.

Using the values of the sine and cosine, ascertain if the following statements are true:

$$\sin^2 40° + \cos^2 40° = 1; \quad \frac{\sin 40°}{\cos 40°} = \tan 40°,$$

$$\sec^2 40° = 1 + \tan^2 40°; \quad \operatorname{cosec}^2 40° = 1 + \cot^2 40°.$$

Write down the percentage error (if any) in each case.

2. Calculate the number of radians in the angle.

3. The sides of a triangle are 7, 10, and 13 respectively, find the sine of the largest angle, and the number of radians in the angle.

4. The sides a, b, c of a triangle are 1·2, 1·6, and 2 feet respectively, find the values of $\sin \frac{A}{2}$, $\cos \frac{B}{2}$, and $\tan \frac{C}{2}$.

Also find the number of degrees and radians in the angle A.

5. In the triangle the sides of which are 7, 12, and 14 feet in length respectively, find the three angles; also show whether the least angle is greater or less than 30°.

6. (i) If $a = 125$, $b = 123$, and $c = 62$, find $\sin A$.

(ii) In the triangle ABC, $b = 215$, $c = 105$, and $A = 74° 27'$, find B and C.

7. If $A = 90°$, $B = 60°$, $C = 30°$, $D = 45°$, show that

$$\sin B \,.\, \cos C + \sin C \,.\, \cos B = \sin A,$$

and that

$$\cos^2 D - \sin^2 D = \cos A.$$

8. Define the sine, cosine, and tangent of an angle. If $\tan A = \frac{1}{2}$, find the values of the following expressions:

(i) $\cos^2 A - \sin^2 A$.

(ii) $\operatorname{cosec}^2 A - \sec^2 A$.

(iii) $\cot^2 A + \sin^2 A$.

9. Make CD, as in Fig. 62, equal to 10 units, and write down the numerical values of $\sin 22\frac{1}{2}°$, $\cos 22\frac{1}{2}°$, and $\tan 22\frac{1}{2}°$.

10. As in the last Example, make DC equal to 10 units, and write down the values of $\sin 67\frac{1}{2}^\circ$, $\cos 67\frac{1}{2}^\circ$, and $\tan 67\frac{1}{2}^\circ$. Verify in each case by reference to Table VI., p. 376, and write down in each case the percentage error.

11. Obtain the numerical values of $\sin 75^\circ$, $\cos 75^\circ$, and $\tan 75^\circ$. Also of $\sin 15^\circ$, $\cos 15^\circ$, and $\tan 15^\circ$. Verify by reference to Table VI., p. 376.

Given the sine of an angle to construct the angle.

Given $\sin \theta = \frac{3}{7}$.

Draw a line AB (Fig. 64), and at point B erect a perpendicular BC to any convenient scale 3 units in length, with C as centre and radius 7 units, describe an arc cutting AB in A, join A to C, then BAC is the angle required. Measure the angle by the protractor, and verify by referring to Table VI.

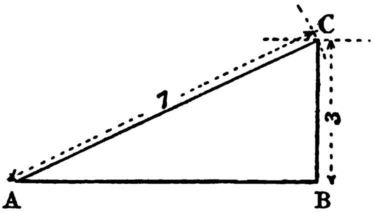


FIG. 64.—Given the sine of an angle to construct the angle.

Given the cosine of an angle to construct the angle.

Given $\cos \theta = \frac{4}{7}$.

Set out $AB = 4$ units (Fig. 65), and erect a perpendicular BC, with A as centre, radius 7 units, describe arc cutting BC in C,

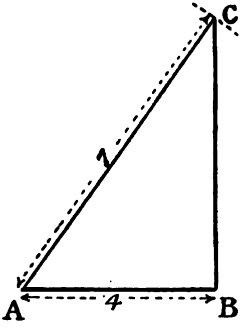


FIG. 65.—Given the cosine of an angle to construct the angle.

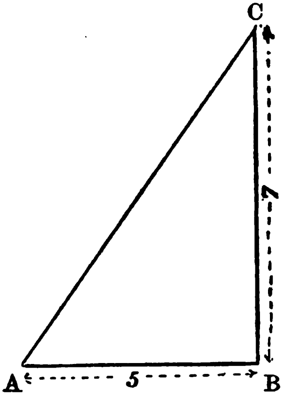


FIG. 66

then *BAC* is the angle required. Measure the angle by the protractor, and verify by Table VI.

Given the tangent of an angle to construct the angle.

Given $\tan\theta = \frac{7}{5}$.

Draw a line *AB*, 5 units in length (Fig. 66), and at *B* draw *BC* perpendicular to *AB* and equal to 7 units; join *A* to *C*; then *BAC* is the angle required. Verify as in the preceding cases.

Inverse Ratios.—A very convenient method of writing $\sin\theta = \frac{3}{7}$ is to write it as $\sin^{-1}\frac{3}{7}$, which is read as the angle whose sine is $\frac{3}{7}$. In a similar manner we may write $\cos\theta = \frac{4}{7}$ and $\tan\theta = \frac{7}{5}$, as $\cos^{-1}\frac{4}{7}$ and $\tan^{-1}\frac{7}{5}$ respectively.

EXERCISES. XLVII.

1. Construct an angle whose sine is $\frac{1}{3}$, and calculate the angle to the nearest second.

2. Show how to construct an angle of 60°, and hence find the numerical values of the sine, cosine, and tangent of an angle of 30°.

3. Construct an angle of 45°, and find from the construction the sine, cosine, tangent, and cotangent of 45°.

4. Show how to construct an angle of 15°, and find from the construction the numerical value of the sine, cosine, and tangent of the angle.

5. The sine of an angle is ·31; find the angle in degrees and radians. Also find the cosine and tangent of the angle.

6. Find the numerical values of the sine and cosine of angles 60° and 75° respectively.

7. If $\cot A = \frac{2}{3}$, find sine A and secant A.

8. Find values of sin 18° and cos 36°.

9. The cosine of an angle is $\sqrt{\frac{2}{3}}$, find the tangent of the angle and the tangent of its complement.

10. Define the sine of an angle, and find the other trigonometrical ratios of the acute and obtuse angle whose sine is $\frac{4}{5}$.

11. Given $a=3$, $c=5$, $B=120°$, find b.

12. Define the cosine, tangent, and cosecant of an angle.

If $\tan\theta = \frac{4}{3}$, find the values of $\sin\theta$, $\sec\theta$, $\sin 2\theta$, $\cos 2\theta$, and $\tan 2\theta$.

13. Show how to construct an angle whose tangent equals 0·9.

14. Define the cosine of an angle, and find the other trigonometrical ratios of the angle whose cosine is $\frac{3}{5}$.

15. (i) If $\tan A$ is $\frac{2pq}{p^2-q^2}$, find $\cos A$ and $\operatorname{cosec} A$.

(ii) Determine A if $p=3{\cdot}24$ and $q=1{\cdot}98$.

(iii) In a triangle ABC where C is a right angle, prove that

$$\tan A+\tan B=\frac{c^2}{ab}.$$

Use of Tables.—As already stated on page 25, the magnitude of an angle is independent of the length of the sides or lines forming the angle; and if at any point B in one of the lines (Fig. 67) a perpendicular BC be drawn meeting the other side in C, the ratio of BC to AC, of AB to AC or BC to AB remains the same for all positions of B along the line AF. Hence, the value of tables in which the numerical values of the sine, cosine, tangent, etc., of an angle for any angle from 0° to 90° are to be found. These values are called the natural sines, cosines, tangents, etc., of the angles. The numerical values can also be obtained by the slide rule (see p. 186).

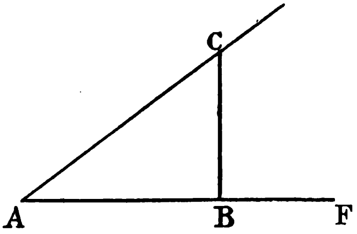


FIG. 67.

Ex. 1. Find the sine of 6° 30′.

First find the number of degrees in the left-hand column, Table VI., then $\sin 6^\circ={\cdot}1045,\ \sin 7^\circ={\cdot}1219$;

difference for 1° or $60'={\cdot}0174$,

and $\frac{30}{60}$ of $\cdot0174=\cdot0087$;

also $\cdot1045+\cdot0087=\cdot1132$;

$\therefore\ \sin 6^\circ 30'={\cdot}1132$.

If the angle had been 6° 33′ we should have found as before

$\sin 6^\circ={\cdot}1045$, difference for $60'={\cdot}0174$;

difference for $33'=\frac{33}{60}$ of $\cdot0174=\cdot0096$;

$\therefore\ 6^\circ 33'={\cdot}1141$.

In the case of the sine and tangent, as the angle increases the numerical values of the sine and tangent increase, but the numerical value of the cosine diminishes as the angle increases.

Hence we have the rule: Find, if necessary, the difference between two consecutive values of the angle and the difference which corresponds to the remaining minutes in a given angle, *add this difference to the number already obtained in the case of the sine and tangent, and subtract it in the case of the cosine.*

Ex. 2. Find the cosine of 6° 33′.

Here

$$\cos 6^\circ = \cdot 9945.$$
$$\text{Diff. for } 33' = \underline{\cdot 0010} \text{ (subtract).}$$
$$\cos 6^\circ 33' = \cdot 9935$$

Tables of Logarithms of Trigonometrical Ratios.—The ratios forming the functions of angles, being numbers, have in each case corresponding logarithms; and as we have already found, these logarithms are in practice much more useful than the numbers themselves.

As the sine and cosine of an angle are never greater than unity, the logarithm of these sines and cosines are negative. In order to avoid the inconvenience of writing and printing negative characteristics, it is found convenient to *add* 10 *to the logarithm of every Trigonometrical Ratio before registering it in the tables.* The numbers thus registered are called *Tabular Logarithms*, and are denoted by prefixing the symbol L. Thus $L \sin A$ refers to the Tabular Logarithm of $\sin A$.

$$L \sin A = \log \sin A - 10.$$

Ex. 1. $L \sin 16^\circ 30'$.

From Table VI.,

$$\sin 16^\circ 30' = \cdot 2840$$
$$\log \sin 16^\circ 30' = \log \cdot 2840 = \bar{1} \cdot 4533$$
$$L \sin 16^\circ 30' = 10 + \bar{1} \cdot 4533 = 9 \cdot 4533.$$

Ex. 2. L cosine 36° 58′.

$$\cos 36^\circ 58' = \cdot 7990$$
$$\log \cdot 7990 = \bar{1} \cdot 9025$$
$$L \cos 36^\circ 58' = 9 \cdot 9025$$

Ex. 3. Take out the logarithm of tan 3° 15′, and calculate the value of the cube root of the tangent.

In Table VI.,

$$\tan 3^\circ = \cdot 0524$$
$$\text{Diff. for } 15' = \underline{\cdot 0044}$$
$$\therefore \tan 3^\circ 15' = \cdot 0568$$
$$\log \cdot 0568 = \bar{2} \cdot 7543.$$

Also $\sqrt[3]{\tan 3^\circ 15'} = \frac{1}{3} \log \tan 3^\circ 15'$

$$= \tfrac{1}{3}(\bar{3} + 1{\cdot}7543)$$

$$= \bar{1}{\cdot}5847.$$

Corresponding Antilog $= {\cdot}3843$.

Hence $\sqrt[3]{\tan 3^\circ 15'} = {\cdot}3843$.

EXERCISES. XLVIII.

1. Take out from Table VI. the logarithm of the tangent of 16° 6′, and calculate the value of the square root of the tangent.

2. Find the logarithm of the tangent of 81° 12′, and the angle whose log cotangent is 9·6172.

Find the numerical value of $\sqrt[3]{(\tan 50^\circ \tan 22^\circ 30')}$.

3. Find log tan 35° 15′, and the numerical value of $\sqrt[3]{(\frac{1}{3} \sin 44^\circ)}$.

4. Find the numerical value of $\sqrt[5]{(\tan 40^\circ \div 65)}$.

5. What is the numerical value of the seventh root of

$$(\tan 53^\circ 30') \div 32\,?$$

6. Find the logarithm of tan 58° 5′; also the value of the cube root of $(\tan 52^\circ 30') \div 15$.

7. Find the logarithm of $(\sin 26^\circ 13')^{-\frac{1}{4}}$.

8. Find the logarithm of $\sin (18^\circ 37')^{-\frac{1}{2}}$.

9. Given that tan 21° 48′ = 0·4, find to three places of decimals the numerical values of tan 68° 12′, cos 21° 48′, and sin 21° 48′.

Applications to Problems on Heights and Distances.—It is not always convenient, or possible, to measure directly the height of a given object, or to find the distance of two objects apart.

Instruments are used for measuring purposes by which the angle between any two straight lines which meet at the eye of the observer can be measured. For this purpose what are called *Sextants* and *Theodolites* are used. By means of these the cross-wires of a telescope can be made with considerable accuracy to coincide with the image of an observed object, and by means of a vernier attached, the readings of the observed angles can be made to a fraction of a minute.

The *angle contained between a horizontal line and the line which meets a given object* is called the *angle of elevation* when the object is above the point of observation; and the *angle of depression* when the object is below.

Thus, if B be the point of observation (Fig. 68) and A the given object, the angle made by the line joining B to A, with the horizontal line BC, is called the *angle of elevation.*

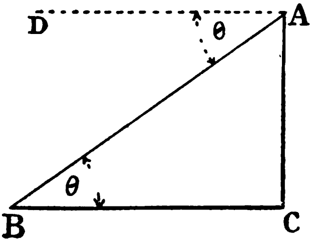


FIG. 68.—Angle of depression and angle of elevation.

In a similar manner if A be the point of observation and B an object, then the angle between the horizontal line (DA, drawn through A) and the line AB is called the *angle of depression.*

In all cases the diagrams made should be as clear and accurate as possible. Even when the result depends alone on calculation it can, and should be, roughly checked by graphic construction.

The following problems will show the methods adopted.

Ex. 1. At a distance of 100 feet from the foot of a tower the angle of elevation of top of tower is found to be 60°. Find the height of the tower.

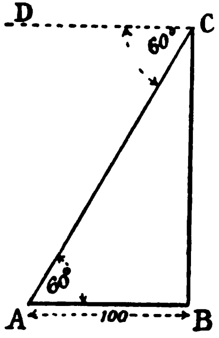


FIG. 69.

To any convenient scale make AB (Fig. 69) the base of a right-angled triangle equal 100 units.

Draw the line AC, making an angle of 60° with AB and intersecting BC at C. Then BC is the required height.

By calculation, $\frac{BC}{AB} = \tan 60°$;

but $\tan 60° = \sqrt{3}$;

$\therefore\ BC = 100\sqrt{3} = 173\ldots$ ft.

Ex. 2. From the top of a tower, the height of which is $100\sqrt{3}$ ft., the angle of depression of an object on a straight level road on a line with the base of the tower is found to be 60°. Find the distance of the object from the tower.

In this case, drawing CD a horizontal line (Fig. 69) through C, the point of observation, making $CB = 100\sqrt{3}$ ft. on any scale, and BA at right angles to CB. Then the point at which a line CA, drawn at an angle of 60° to the horizontal line DC, meets BA gives the distance BA required $= 100$ ft.

EXERCISES. XLIX.

1. The angle of elevation of the top of a steeple is 45° from a point in the same horizontal plane as its base, and is 30° from a point 30 feet directly above the former point; find the height and distance of the steeple.

2. A pole is fixed on the top of a mound, and the angles of elevation of the bottom and top of the pole are 30° and 60° respectively; prove that the height of the pole is twice the height of the mound.

3. The altitude of a tower, observed at the end of a horizontal base of 100 yards measured from its foot, is 30°; find its height.

4. From the top of a tower 150 feet high the angles of depression of the top and bottom of a vertical column, standing on the same horizontal plane, were observed to be 22° 15′ and 44° 30′; find the height of the column.

5. A line AB, 450 yds. long, is measured close by the brink of a river, and a point C, close to the bank of the river on the other side, is observed both from A and B; the angle CAB is 52°, and CBA is 70°; find the width of the river.

6. A tower on the bank of a river is 200 feet high, and carries a flag-staff 30 feet high. A man 6 feet in height stands at the bottom of the tower. To an observer on the opposite bank the man and the flag-staff subtend equal angles. Find the width of the river.

7. From the lower window of a house the angle of elevation of a church tower is observed to be 45°, and from a window 20 feet above the former, 40°; how far is the house from the church?

8. Looking down from the top of a hill, the foot of which is 125 feet below the level of his eye, a man sees a statue 40 feet high, and the column, 60 feet high, upon which it stands, subtending equal angles at his eye. How far off is he in a horizontal direction from the object?

9. What is the angle of depression of an object? From the top of a hill the angles of depression of two consecutive mile-stones on a straight, level road, were found to be 12° 13′ and 2° 45′ respectively; find the height of the hill.

10. The angle of elevation of the top of a hill is observed to be 5°; after walking one mile directly towards the hill the angle of elevation is 14° 30′. Find the height of the hill.

11. An object 10 ft. high is placed on the top of a tower, and subtends an angle of 6° at a place which is in the same horizontal plane as the foot of the tower, and is 50 ft. distant from it. Determine the height of the tower.

12. From the top of a tower by the seaside, 150 ft. high, it was found that the angle of depression of a ship's hull was 36° 18′. Find the distance of the ship from the foot of the tower.

13. ABC is a triangle with a right angle at C, CB is 30 feet long, and BAC is 20°. If CB be produced to a point P, such that PAC is 55°, calculate the length of CP.

14. The angular height of a tower is observed from two points A and B 1000 feet apart in the same horizontal line as the base of the tower. If the angle at A is 20° and at B 55°, find the height of the tower.

15. The angle of elevation of the top of a steeple is 30°. If I walk 50 yards nearer, the angle of elevation becomes 60°. What is the height of the steeple?

16. The elevation of a tower from a point A due N. of it is observed to be 45°, and from a point B due E. of it to be 30°. If $AB=240$ feet, find the height of the tower.

17. If from a point at the foot of the mountain, at which the elevation of the observatory on the top of Ben Nevis is 60°, a man walks 1900 feet up a slope of 30°, and then finds that the elevation of the observatory is 75°, show that the height of Ben Nevis is nearly 4500 feet.

18. The angular elevation of the top of a vertical pole changes from 45° to 30° as an observer moves 100 feet away from the pole in a horizontal line through the pole. Find the height of the pole, the observer's eye being $5\frac{1}{2}$ feet above the ground.

19. From a point P on the bank of a river just opposite a post Q on the other bank, a man walks at right angles to PQ to a point R, such that PR is 100 yards; he then observes the angle PRQ to be 32° 17′; find the breadth of the river, having given $\tan 32° 17' = \cdot 6318$.

20. A man wishes to find the height of a church spire which stands on a horizontal plane; at a point on this plane he finds the angle of elevation of the top of the spire to be 45°; after walking 100 feet toward the spire he finds the angle of elevation to be 60°; what is the height of the spire?

21. A and B are two points on one bank of a straight river and C a point on the opposite bank; the angle BAC is 30°, the angle ABC is 60°, and the distance AB is 400 feet; find the breadth of the river.

22. Find the height of a chimney which is such that on walking towards it 100 feet in a horizontal line through its base, the angular elevation of its top changes from 30° to 45°.

23. The angles of elevation of a spire at two places due east of it and 200 feet apart are 45° and 30°; find the height of the spire.

24. A and B are two hill-tops 34,920 feet apart, and C is the top of a distant hill. The angles CAB and CBA are observed to be 61° 53′ and 76° 49′ respectively. Prove that the distance from A to C is 51,515 feet.

25. From the top of a cliff 1000 feet high, the angles of depression of two ships at sea are observed to be 45° and 30° respectively; if the line joining the ships points directly to the foot of the cliff, find the distance between the ships.

Summary.

All triangles consist of six parts, three angles and three sides; the sum of the three angles of a triangle is 180°.

Right-angled triangles.—If one angle is a right angle or 90°, the sum of the other two angles is 90°; each of these latter angles being less than 90° is called an acute angle, and in all right-angled triangles having the same acute angle the ratio of the sides is the same. These ratios are known as:

The **sine.**—The sine of an angle is formed by the ratio of the side opposite the angle to the hypothenuse.

The **cosine.**—The ratio of the side adjacent to the angle, to the hypothenuse.

The **tangent.**—The ratio of the side opposite the angle to the side adjacent.

The reciprocals form three other ratios. Thus the reciprocal of the sine is called the *cosecant.* The reciprocal of the cosine the *secant.* The reciprocal of the tangent the *cotangent.*

When the acute angles are 60° and 30°, the sides of the triangle are proportional to 1, 2, and $\sqrt{3}$.

When the acute angles are each 45°, sides are as 1 to $\sqrt{2}$.

The **complement** of an angle is the remainder after subtracting the angle from 90°.

The **supplement** is the remainder after subtracting the angle from 180°.

$$\sin^2 A + \cos^2 A = 1\ ; \quad \sec^2 A = 1 + \tan^2 A\ ;$$
$$\operatorname{cosec}^2 A = 1 + \cot^2 A.$$

CHAPTER XII.

LABOUR-SAVING METHODS. THE SLIDE RULE.

Slide Rule.—It will be already clear to the reader who has followed the section dealing with logarithms, that by their use multiplication of two or more numbers is effected by adding the logarithms of the factors, and division by the subtraction of the logarithms of the factors. Or, shortly, by the use of logarithms multiplication is replaced by addition, and division by subtraction. Hence if instead of the equal divisions of a scale (Fig. 70) unequal divisions corresponding to the logarithms were employed, then, when performed graphically, multiplication will correspond to addition and division to subtraction.

It is, as has been seen, an easy matter to add together two linear dimensions by means of an ordinary scale or rule. Thus, to add 2 and 3 units together. Assume the scale *B* (Fig. 70) to slide along the edge of the scale *A*, then the addition of the numbers 2 and 3 is made when the 2 on *B* is coincident with 0 on *A*; the addition of the two numbers is found to be 5 opposite the number 3 on the scale of *A*.

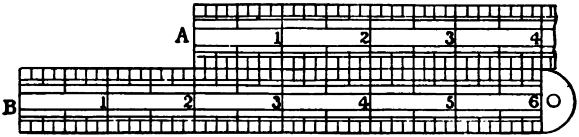


FIG. 70.

If the scales on *A* and *B* were not divided in the proportion of the numbers, but of the logarithms of the numbers, then by this system of graphic logarithms, by sliding one scale along the other in the manner described, addition would be performed, but, as the scales are logarithmic, this would correspond to the product of the numbers added.

Similarly, the number corresponding to the difference would be the quotient.

This graphic method was first put to practical use by Edmund Gunter, who invented in 1614 what was called the "Line of Numbers," a compact and handy logarithmic table, by means of which the arithmetical operation of multiplication, division, involution, and evolution could be performed by inspection.

For similar calculations **Napier** invented sets of handy tables or arithmetical operations called "Napier's Bones," which consisted of sets of numbers engraved on thin slips of brass, ivory, wood, bone, etc., usually made in a rectangular form and of small size, viz. 2 inches in length by ¼ inch in width.

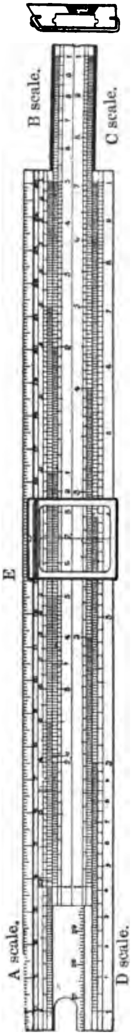


FIG 71.—Slide Rule.

Construction of Slide Rule.—As already stated the object of the slide rule is to perform *arithmetical calculations* in a simple manner.

There is a great saving of time and labour effected by its use, as it solves at sight all questions depending on ratio.

It consists of a fixed part or rule containing a groove in which a smaller rule slides.

Reference to Fig. 71 shows that the upper part of the rule contains two scales exactly alike, while the lower part of the rule contains only one scale, its length double that of the upper one. As the upper part contains two scales, it will be convenient to refer to the division 1 in the centre of the rule, shown at *E*, as the *left-hand* 1, the other to the right of it as the *right-hand* 1.

There are two scales on the smaller rule or *slide*, as we may call it, at *B*; and at *C* one double the length. Hence, the scales on the slide correspond to those on the rule.

It will be convenient to refer to the four scales by the letters *A*, *B*, *C*, *D* as shown in Fig. 71.

If an examination of a slide rule be made it will be found to consist of a fixed part or frame, a slide, and, in the case of a Gravêt slide rule, there is also an additional movable frame or thin metal *cursor*, held in position on the face of the rule by sliding in two grooves. This is shown at *E* and in the end view. Although it slides freely along the instrument, any shake which might otherwise occur is prevented by a small steel spring placed at the upper part of the carrier.

The principle of action is the same in all slide rules, although the arrangement of the lines depends upon the purpose to which the rule is to be applied. The modified form of the slide rule, or Gravêt rule, which we propose to explain, is one of the most accurate instruments of the kind that can be obtained. The instrument, with the exception of the cursor *E*, is usually made of boxwood or mahogany. The wood is faced with white material, as the black division lines showing more clearly on the white background.

Graduation of Slide Rules.—In Fig. 71, which shows a "Gravêt" rule, it will be seen that the distance apart of the divisions are by no means equal. The divisions and subdivisions are not equidistant as in an ordinary scale, but are logarithms of the numbers, and are set off from the left or commencing unit.

In studying Indices we have already found that

if 10^3 be multiplied by 10^4 the result is 10^{3+4} or 10^7.

Also $10^{\cdot 25} \times 10^{\cdot 3} = 10^{\cdot 25+\cdot 3} = 10^{\cdot 55}$.

„ $10^{\cdot 3010} \times 10^{\cdot 4771} = 10^{\cdot 3010+\cdot 4771} = 10^{\cdot 7781}$.

From the definition of a logarithm,

2 is the logarithm of 100, since $10^2=100$.

Or, as 10 raised to the power 2 gives 100, the logarithm of 100 is 2.

In a similar manner if 10 be raised to a power ·4771, we obtain a result 3; $\therefore \log 3 = \cdot 4771$.

Also, since $10^{\cdot 3010}=2$; $\therefore$ $\log 2=\cdot 3010$.

Hence ·7781 is the log of $6=10^{\cdot 3010+\cdot 4771}$.

Again $$\frac{10^5}{10^2}=10^{5-2}=10^3.$$

So also $$\frac{10^{\cdot 7781}}{10^{\cdot 3010}}=10^{\cdot 7781-\cdot 3010}=10^{\cdot 4771}.$$

Or more simply, to divide 6 by 2,

$$\log 6-\log 2=\cdot 7781-\cdot 3010$$
$$=\cdot 4771\,;$$

and ·4771 is the log of 3.

Hence $$\frac{6}{2}=3.$$

Simple exercises similar to the above will be found very useful as a first step, and such practice will enable the student to handle the numbers with certainty and ease. The student should make a slide rule, using two strips of cardboard or thick paper.

Assuming any length, such as from 1 to *E*, scale *A* to be divided into 10 equal parts, then the distance from 1 of any intermediate number from 1 to 10 is made proportional to its logarithm.

To find the position of the 2nd division, since $\log 2=\cdot 301$, ·301 parts would indicate its position.

In like manner the 3rd division would be ·477 parts; the 4th, ·602 parts; the 5th by ·699 parts.

Hence, from	1	to	2	contains	·301	of those parts,	or is	log 2.
„	2	„	3	„	·477	„	„	log 3.
„	3	„	4	„	·602	„	„	log 4.
„	4	„	5	„	·699	„	„	log 5.
„	5	„	6	„	·778	„	„	log 6.
„	6	„	7	„	·845	„	„	log 7.
„	7	„	8	„	·903	„	„	log 8.
„	8	„	9	„	·954	„	„	log 9.

Denoting the distance of any division from point 1 by x.

If l denote the length of the scale from 1 to *E*, and L the logarithm of the number indicating the division required, then

$$x=l\,.\,L.$$

Ex. 1. To construct a scale of 10 inches in length, it is necessary to determine the distance of each division of the scale from 1. This is effected by multiplying the logarithms of all the numbers from 1 to 10 by 10. $\therefore x=10L$.

The excellence of any slide rule depends upon the skill with which these division lines have been constructed, so that they are as accurate as it is possible to make them, and in dealing with a carefully made slide rule we deal with the *effect* of a considerable amount of labour and thought which have been expended in its construction.

Although a knowledge of logarithms is not essential before a slide rule is used, any more than it is necessary that a man should be able to make a watch before he is allowed to use one, or that he should understand the nature of an electric current before using an electric bell, it is much better where possible to clearly understand the principles underlying the construction of any instrument.

Multiplication.—In (Fig. 71) putting the units figure of the slide opposite the 2 on the fixed scale A, we get registered the products of all the numbers on the slide and 2 above. Thus $2\times1=2$, $2\times2=4$, $2\times3=6$, etc. The units figure may denote 1, or 10, or 100, etc.; thus the products may be read off as $2\times10=20$, $2\times20=40$, or $2\times100=200$, etc. Or we may use the lower scale D of the rule and scale C of the slide. Always make 1 of slide coincide with either of the factors on the scale, and the product will be found on the scale opposite to the other factor read on the slide. If the product cannot be found when the left-hand 1 is used, the right-hand 1 must be used.

Facility in using the rule can only be obtained by practice, and as soon as the principle is understood the so called rules are not of much use.

From any table of areas and circumference of circles, examples for practice can be obtained, or the examples given on p. 246 may be used for the same purpose, it is not necessary to give such at this stage.

Ex. 2. Multiply 12·8 by 4.

Setting the 1 on the slide to 12·8 on the rule, then opposite 4 on slide is the number 51·2.

Ex. 3. The diameter of a circle is 2·5 inches, find its circumference.

Set 1 on slide to 2·5 on scale *A* opposite 3·14 (which is marked on slide), is the circumference 7·85 inches.

Division.—Set the divisor on *B* under the dividend on *A*, and read the quotient on *A* over index of *B*, *or* set divisor on *C* over dividend on *D*, and read the quotient on *D* under the units figure of *C*.

Ex. 1. Divide 9 by 4.

Setting the 4 on scale *B* opposite 9 on scale *A*, then coincident with 1 is the answer 2·25.

Ex. 2. Divide 9·5 by 2·5.

Set 2·5 on scale *C* coincident with the division 9·5 on *D*, then coincident with 1 on *C* is the answer 3·8 on *D*.

Ex. 3. The circumference of a circle is 29·53 inches, find the diameter.

Setting 3·14 on slide opposite 29·53, coincident with 1, we read 9·4, the answer required.

Proportion.—Making use of the two upper scales *A* and *B*, operate so as to find the quotient, and *without reading off the answer*, look along the rule for the product of the quotient by the third factor in the proportion.

Ex. 1. $3 : 4 :: 9 : x$.

Read off the answer 12 by the process described, or put the proportion in the form of ratios; thus $\frac{3}{4}=\frac{9}{x}$.

Place 4 on *B* under 3 on *A*, and under 9 on *A* read off the answer $x=12$ on *B*.

Involution.—On inspection, the numbers on the upper scale *A* are seen to be the squares of the numbers on the lower scale.

To obtain the square of a fractional number some difficulty would be experienced in noting coincidence of divisions on *A* and *D*, separated as they are by the slide, in this case we can make use of the *cursor*, thus:

Set the cursor to coincide with the given number on the scale *D*, and by it read off the square of the number on scale *A*. In this manner, the fourth, or any even power, can be obtained.

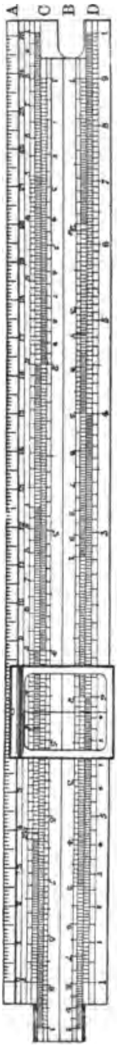


FIG. 72.—Slide rule, slide inverted for involution and evolution.

Square Root.—In extracting square roots one reverses the above process.

The number, the root of which is required, is found on the scale *A*, and its root is found directly below, as before. The cursor enables coincidence of the two divisions denoting the number and its root to be readily obtained.

As shown on p. 244, the area of a circle is $3{\cdot}1416 \times r^2$, or $\cdot 7854d^2$, where r is the radius and d the diameter of the given circle. Conversely, if the area of a circle is given, the diameter can be obtained from :

$$\text{diameter} = \sqrt{\frac{\text{area}}{\cdot 7854}}.$$

Ex. 1. Find area of a circle 3″ diameter.

The mark on the cursor is set to the 3 on the lower scale ; the upper mark registers 9. Then moving the slide to the right until the 1 coincides with the mark, we have coincident with ·7854 (which is marked on the scale) the required area 7·01 square inches.

Ex. 2. Find area of a circle 2·5″ diameter.

The square of 2·5 is seen to be 6·25, and multiplied by ·7854 the area is 4·9 square inches.

Practice in Use of Slide Rule.—Practice in the use of the slide-rule obtained by getting the powers and roots of various numbers ; by determining areas of circles from given diameters, and conversely, diameters from given areas. The circumferences and diameters may be treated in like manner. All results when obtained may be verified by calculations, or from tables.

To obtain the cube of a number. Bring the right-hand 1 of scale *C* to the given number on *D*. Then over the same number on the scale *B* read off the required cube on *A*.

Instead of the above plan the slide may be inverted (Fig. 72), keeping the same face upwards. The scale *B* will now move along scale *D*. Put in coïncidence on the scales *B* and *D* the two marks indicating the number the cube of which is required, then opposite the right-hand 1 on the slide the cube required will be found on scale *A*.

Cube Root.—As in the case of finding the cube of a number, two methods may be used to obtain the cube root of a number.

First Method.—Move the slide to right or left until under the given cube on *A* is found on *B* a number identical with a number which is simultaneously found on the *D* scale coincident with the 1 on scale *C*.

Second Method.—The inverted slide is used. This is placed with the 1 of the slide coincident with the number the cube root of which is required. Then find what number on the scale *D* coincides with the same number on the inverted scale of *B*; this number is the cube root required.

It will be found advisable to obtain the cube roots of easy numbers before proceeding to more difficult cases.

Ex. 1. Find the cube root of 64.

Move the slide from right to left, and it will be found that 4 on scale *B* coincides with 64 on scale *A*, simultaneously with 4 on *D* and 1 on *C*. Hence 4 is the cube root required.

Invert the slide, keeping the same face uppermost. Set 1 on *C*, inverted to 64 on scale *A*, the division on *B* which coincides with *D* is 4. Hence 4 is the cube root.

2. Find the areas of circles the diameters of which are 2, 3, 4, 6, 2·3, 3·2, 4·2, and 6·2 inches respectively.

3. Find the area of an air-pump bucket if its area is $\frac{1}{8}$ that of a cylinder of 63 inches diameter.

4. Find the diameter of a cylinder, the area being 20 square inches.

5. Write down the square and cube roots of the numbers 1, 2, 3, ... 10 inclusive.

Logarithms of Numbers, Lines, and Tangents.—On the reverse side of the slide (Fig. 73) the value of sines are to be found on the upper edge, marked s, tangents on the lower, and logarithms in the middle. Moreover, on the reverse side of the rule a gap is found having its edges bevelled, and on it two transverse marks. Logarithms can be obtained by using the lower mark, and sines by using the upper one. Tangents are read by using the lower mark and a corresponding one on the opposite end of the rule. The values of the sines and tangents may also be obtained by reversing the slide and reading direct.

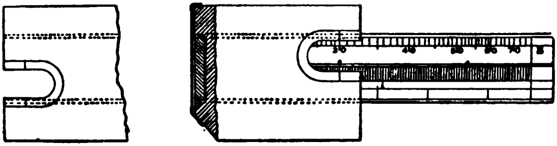


Fig. 73.—Showing the reverse side of the slide-rule.

To obtain log 2, set the 1 on the slide to coincide with 2 on lower scale *B*, reverse the rule, and opposite the lower mark, the log of 2 = ·301 is read off; also setting it opposite 3, log 3 = 477, etc.

Ex. 1. Obtain and write down the logarithms of all numbers from 1 to 10. Verify the results by reference to Table IV., p. 372.

Functions of Angles.—Given the numerical value of the sine or tangent of an angle, the number of degrees in the angle can be found; or, conversely, given the number of degrees in the angle, the numerical value can be ascertained:

Ex. 1. To find the numerical value of sin 30°.

Reverse the rule as shown in Fig. 73, placing sin 30° opposite upper mark.

Again reverse the rule, and opposite right-hand 1 on scale *A* the numerical value ·5 on slide is obtained on scale *B*.

Similarly, placing sin 60° opposite the mark, the value ·866 is obtained.

Also, sin 50° = ·766 may be read off.

For practice other values should be selected, their numerical values written down and afterwards verified.

Ex. 2. Obtain and write down the numerical values of the sines of angles of 15°, 20°, 25°, 30°, and for intervals of 5° up to sin 70°.

Conversely, when the numerical value of the sine is given, the corresponding angle can be found.

Tangents.—At the opposite end of the rule a gap similar to the one just described is to be found; this may be used to find the numerical value of tangents.

Ex. 3. Find the value of tan 30° and tan 20°.

Move slide until 30° is opposite the mark, then on upper scale coincident with 1 the value ·577 is obtained.

In like manner tan 20° = ·364.

Ex. 4. Obtain and tabulate the numerical values of the tangents of angles from 10° to 40°.

Ex. 5. Write down the values of the sine and cosine of 5°, 10°, etc., up to and including 45°. Verify by reference to Table VI., p. 376.

Fuller's Slide Rule, or calculating slide scale, is a handy compact form of slide rule, equal to a slide 83 feet long. The ordinary process of multiplication, division, involution, and evolution can easily be performed on it.

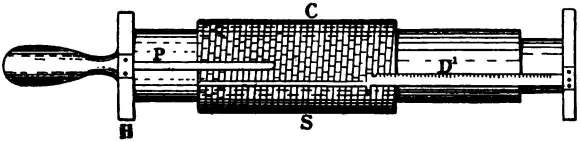


FIG. 74.—Fuller's slide-rule.

It consists, as shown in Fig. 74, of a cylinder *C*, having a single logarithmic scale *S* graduated round it in a spiral direction.

The cylinder can be turned round or moved up and down a sleeve, which is attached to the handle *H*. The readings are obtained by two pointers as shown, one of which is attached to the handle *H*, the other to a smaller cylinder which can slide easily in the sleeve. As the cylinder must be moved for every result, it is not very convenient when required to multiply or divide by a constant, but is very useful for ordinary arithmetical computations. The results can be read off to four places of decimals.

To use the instrument say to multiply $4 \times 12{\cdot}5 \div 5{\cdot}6$. Set the pointer P (fixed to the handle) to the first number 4. Then bring the movable point D' to the denominator 5·6, next move the cylinder C until the second figure 12·5 appears under the movable pointer D', and finally read the answer on the cylinder C underneath the fixed pointer P.

Circular Slide Rules.—So-called slide rules or pocket calculators, in shape and size similar to a watch, can be obtained. On account of their shape they are very convenient and portable, and are capable of performing all the processes of an ordinary slide rule.

CHAPTER XIII.

ARITHMETICAL PROGRESSION. GEOMETRICAL PROGRESSION. GEOMETRIC MEAN AND ARITHMETIC MEAN. SIMPLE INTEREST. COMPOUND INTEREST. DISCOUNT.

Arithmetical Progression.—Quantities are said to be in arithmetical progression when they *increase* or *decrease* by the *addition* or *subtraction* of the same quantity. Thus the numbers 1, 2, 3, 4, which increase by the addition of 1 to each successive term; 21, 18, 15, 12, which decrease by the subtraction of 3 from each successive term; a, $a+d$, $a+2d$, etc.; and a, $a-d$, $a-2d$, increasing or diminishing by the addition or subtraction of a quantity d—are in arithmetical progression. It will be seen that if a be the first term, $a+d$ the second, $a+2d$ the third, etc., any term, such as the eighth, is equal to a added to d repeated $(8-1)$ times, or $a+7d$; therefore, if l be the nth or last term,

$$l=a+(n-1)d. \quad\text{.............................(i)}$$

If s denote the required sum, then using the same letters as before,

$$s=a+(a+d)+(a+2d)+\ldots+(l-2d)+(l-d)+l;$$

also writing the series in the reverse order,

$$s=l+(l-d)+(l-2d)+\ldots+(a+2d)+(a+d)+a.$$

Adding these two series together, we obtain

$$2s=(a+l)+(a+l)+\ldots\text{ to } n \text{ terms}$$

$$=n(a+l).$$

$$\therefore\ s=\frac{n}{2}(a+l), \quad\text{...(ii)}$$

or in words, *the* **sum** *of a number of terms in arithmetical progression is found by multiplying the sum of the first and last terms by half the number of terms.*

In this form, when the first, last, and number of terms are known, the sum can be found.

The sum can also be obtained when the first term and the number of terms are given, as follows:

From (i) $l=a+(n-1)d.$

Substituting this value in (ii),

$$s=\frac{n}{2}\{2a+(n-1)d\}.$$

Ex. 1. Find the 11th term of the series 1, 3, 5, etc.

Here $a=1,\ d=2,\ n=11;$

$\therefore$ the 11th term $=1+(11-1)2=21.$

Ex. 2. Find the sum of the series 1, 3, 5, etc., to 120 terms.

$$\begin{aligned} s&=\tfrac{120}{2}\{2\times 1+(120-1)2\}\\ &=240\times 60=14400. \end{aligned}$$

Ex. 3. Find the sum of the series 15, 11, 7, 3 − 1, − 5, to 20 terms.

$$\begin{aligned} s&=\tfrac{20}{2}\{2\times 15, +(20-1)-4\}\\ &=-46\times 10=-460. \end{aligned}$$

Ex. 4. Find the sum of $\frac{1}{3}$, $\frac{2}{3}$ ... to 150 terms.

Here $a=\frac{1}{3},\ d=\frac{1}{3};$

$$\therefore\ s=\tfrac{150}{2}\left\{2\times\tfrac{1}{3}+(150-1)\tfrac{1}{3}\right\}=3775.$$

Arithmetic Mean.—The middle term of any three quantities in an arithmetical progression is the *arithmetic mean* of the other two.

Thus if a and b are the two quantities, and A the arithmetic mean, then a, A, and b form three terms of an arithmetical progression, and

$$\therefore\ A-a=b-A,$$

or

$$A=\frac{b+a}{2}.$$

Hence the arithmetical mean of two quantities is half their sum.

It is always possible between any two given quantities to insert a number of terms such that the whole series are in arithmetical progression.

Thus from (i) $d=\dfrac{l-a}{n-1}$ gives the common difference.

Ex. 1. **Insert 6 arithmetic means between 1 and 43.**

Including the two given terms, the number of terms will be 8. Hence, we have to find an arithmetical progression of which 1 is the first term and 43 the last.

$$\therefore \ d=\frac{43-1}{8-1}=6.$$

The required means are 7, 13, etc.

EXERCISES. L.

1. The 1st term is 5 and the 7th is 23, find the 20th.

2. The sum of 20 terms is 500, and the last term is 45, find the 1st term.

3. Sum the following series to 12 terms:

$$1+\tfrac{6}{5}+\tfrac{7}{5}+\text{etc.}$$

4. Find the sum of the following series:

$$4,\ 3\tfrac{1}{4},\ \text{etc., to 20 terms.}$$

$$\tfrac{1}{3}-\tfrac{1}{3}-1-\text{etc., to 20 terms.}$$

5. Find the sum of $\frac{1}{4}-\frac{1}{4}-\frac{3}{4}-\ldots$ to 21 terms.

6. The 1st term is 3 and the 3rd term 9, find the sum of the first 20 terms.

7. The sum of the first 7 terms is 49, and the sum of the next 8 is 176. What is the series?

8. The 3rd term is 7 and the 7th is 3. What is the series?

9. Find the 30th term, the 1st term being 17 and the 100th term -16.

10. If the 1st term is 50 and the 5th term 42, find the sum of 21 terms.

11. The sum is 1625, the 2nd term 21, and the 7th term 41; find the number of terms.

12. If the 20th term is 15 and the 30th term is 20, find the sum of the first 25 terms.

13. How many strokes are struck in a day by a clock that tolls the hours?

Sum the series:

14. 14·2, 12·3, 10·4, etc., to 15 terms.

15. 13·1, 11·4, 9·7, etc., to 15 terms.

16. $2+3\frac{1}{3}+4\frac{2}{3}+$ etc. to 10 terms.

17. $64+96+128+$ etc. to 16 terms.

18. 3, $3\frac{1}{2}$, 4, etc., to 10 terms.

19. $-2\frac{1}{2}$, -2, $-1\frac{1}{2}$ to 21 terms.

20. $1\frac{2}{3}$, 2, $2\frac{1}{3}$ to 8 terms.

Sum the series:

21. $-1\frac{1}{3}, -1, -\frac{2}{3}$ to 17 terms.

22. $1{\cdot}2 - 2{\cdot}1 - 5{\cdot}4$ to 10 terms.

23. The sum of n terms of the series 2, 5, 8, etc., is 950; find n.

24. Sum $7 + 32 + 57 +$ etc. to 20 terms.

Geometrical Progression.—Quantities which increase or decrease by a *constant factor* are said to be in *geometrical progression.*

Thus, 1, 3, 9, 27, etc.; 4, 2, 1, $\frac{1}{2}$, etc.; a, ar, ar^2, etc.; are quantities respectively in geometrical progression. The constant factors 3, $\frac{1}{2}$, and r respectively are called the *common ratios*, and may be *found by dividing any term by the term preceding it.*

If we examine the series a, ar, ar^2, etc., where a is the first term and r the common ratio, it is obvious that any term, such as the 6th, is equal to a multiplied by r raised to the power $(6-1)$, or is equal to ar^5.

Hence if l denote the last or nth term,

$$l = ar^{n-1}. \quad \text{.................................(i)}$$

If s denote the sum of n terms, then

$$s = a + ar + ar^2 + \ldots + ar^{n-2} + ar^{n-1}. \quad \text{..................(ii)}$$

Multiplying every term by r, we obtain

$$rs = ar + ar^2 + ar^3 + \ldots + ar^{n-2} + ar^{n-1} + ar^n. \quad \text{..........(iii)}$$

Hence, subtracting (ii) from (iii),

$$rs - s = ar^n - a,$$

or

$$s(r-1) = a(r^n - 1);$$

$$\therefore \quad s = \frac{a(r^n - 1)}{r - 1}. \quad \text{............................(iv)}$$

By changing the signs in both numerator and denominator, (iv) may be written

$$s = \frac{a(1 - r^n)}{1 - r}.$$

Ex. 1. Sum the series 1, 3, 9, etc., to 12 terms.

The common ratio is 3. Hence by (iv),

$$s = \frac{1 \times (3^{12} - 1)}{3 - 1} = \frac{(3^{12} - 1)}{2}$$
$$= 265720.$$

Note.—3^{12} is most easily obtained by using logarithms; obtain 3^{12} in this way, subtract unity from it, and divide the difference by 2.

Ex. 2. Sum the series $1+\frac{2}{3}+\frac{4}{9}$, etc., to 10 terms.

$$s=\frac{1-\left(\frac{2}{3}\right)^{10}}{1-\frac{2}{3}}=\left\{1-\left(\frac{2}{3}\right)^{10}\right\}3$$

$$=\frac{174075}{59049}=2{\cdot}95.$$

Geometric Mean.—The middle term of any three quantities in a geometrical progression is said to be the *geometric mean* of the other two, and is written G.M.

To insert a given number of geometric means between two given quantities.

From $l=ar^{n-1}$, we obtain

$$r^{n-1}=\frac{l}{a}.$$

Ex. 1. **Insert three geometric means between 2 and 32.**

Including the two given terms the number of terms will be 5. Hence we have 5 numbers in a geometrical progression, the first 2 and the last 32.

$$\therefore \quad r^{5-1}=\frac{32}{2};$$

$$\therefore \quad r^4=16;$$

$$r=2.$$

Hence the means are 4, 8, 16.

We have found that the sum of n terms is given by

$$s=\frac{a(1-r^n)}{1-r}.$$

Now if r be a proper fraction, it is evident that r^n decreases as n increases.

Thus let $r=\frac{1}{10}$, then $r^2=\frac{1}{100}$, $r^3=\frac{1}{1000}$, etc.; hence by making n sufficiently great, the sum of n terms differs from $\frac{a}{1-r}$ by as small a quantity as desired.

This is perhaps better expressed by writing

$$s=\frac{a(1-r^n)}{1-r}=\frac{a}{1-r}-\frac{ar^n}{1-r},$$

and in the limit when n is indefinitely great, the value of $\frac{ar^n}{1-r}$ is zero.

Hence the sum of n terms is $\frac{a}{1-r}$; this is usually stated thus: *The sum of an infinite number of terms of a geometrical progression in which the common ratio is a proper fraction is* $\frac{a}{1-r}$.

Ex. 1. Find the sum of an infinite number of terms of the series $\frac{1}{5}+\frac{1}{25}+\frac{1}{125}$, etc.

$$s=\frac{\frac{1}{5}}{1-\frac{1}{5}}=\frac{1}{4}.$$

Ex. 2. Find the value of $\cdot\dot{3}$.

This is represented by $\frac{3}{10}+\frac{3}{100}+\frac{3}{1000}+$ etc.

$$\therefore\ s=\frac{\frac{3}{10}}{1-\frac{1}{10}}=\frac{3}{10-1}=\frac{1}{3}.$$

EXERCISES. LI.

1. Find the 6th term and the sum of 6 terms of the series 2, 6, 18, etc.

2. Find the 6th term and the sum of 6 terms of the series 3, 6, 12.

3. The 5th term is 81 and the 2nd 24, find the series.

4. Find the sum to 12 terms of $1-\frac{5}{6}+\frac{25}{36}-$ etc.

5. The 1st and 2nd terms are 3, -4; find the 6th term and the sum of the first 10 terms.

6. If the 1st term is 3 and 3rd term 12, find the sum of 8 terms.

7. Sum to 11 terms, $-\frac{1}{2}+\frac{1}{3}-\frac{2}{9}+$ etc.

8. Sum to 21 terms, $\frac{1}{4}-\frac{1}{4}-\frac{3}{4}$, etc.

9. Find the series when the 5th term is 81 and the second 24.

10. The sum of the first 6 terms is 9 times the sum of the first 3 terms; find the common ratio.

Sum to infinity the following series:

11. 9·6, 7·2, 5·4, etc.

12. $\frac{1}{2}+\frac{1}{3}+\frac{2}{9}$, etc.

13. $4-3+\frac{9}{4}$, etc.

14. $\cdot 8-\cdot 64+$ etc.

15. $7-\frac{7}{4}+\frac{7}{16}$, etc.

16. $\cdot\dot{7}$.

Simple Interest.—Money which is paid for the loan of a given sum of money, borrowed for a certain time according to a fixed rate, is called *interest.*

The sum borrowed on which the interest is reckoned is called the *principal.*

The sum per cent. per annum, or the interest of £100 for one year, is called the *rate per cent. per annum*, or simply the rate per cent.

Thus, when the interest is £4 for every £100 for a year, the rate is 4 per cent.

Interest may be *simple* or *compound*; in the first the principal remains unaltered, in the second the principal is periodically increased.

The *principal*, together with the *interest* in any given time, is called the **amount** in that time.

In simple interest the relation is

$$\text{interest} = \frac{\text{principal} \times \text{rate} \times \text{time}}{100},$$

the time being expressed in years.

Ex. 1. Find the interest of £422. 10s. for 4 years 6 months at $3\frac{1}{2}$ per cent.

$$\text{Interest on £100 for one year} = £3\tfrac{1}{2}.$$

$$\text{,,} \quad £1 \quad \text{,,} \quad = £\frac{3\frac{1}{2}}{100}.$$

$$\text{,,} \quad £422.\ 10\text{s.} \quad \text{,,} \quad = \frac{422\frac{1}{2} \times 3\frac{1}{2}}{100}.$$

$$\therefore \text{ Interest for } 4\tfrac{1}{2} \text{ years} = \frac{422\frac{1}{2} \times 3\frac{1}{2}}{100} \times 4\tfrac{1}{2}$$

$$= £66.\ 10\text{s.}\ 10\tfrac{1}{2}\text{d.}$$

These multiplications and divisions are readily performed by logarithms.

EXERCISES. LII.

1. Find the simple interest on £1320 for $6\frac{1}{2}$ years at $3\frac{3}{4}$ per cent. per annum.

2. Find the simple interest on £275 for 8 years at $3\frac{1}{4}$ per cent. per annum.

3. Find the amount of £3750 in 4 years at $4\frac{1}{2}$ per cent.

4. In how many years will £4350 amount to £5573. 8s. 9d. if put out at $3\frac{3}{4}$ per cent.?

5. In what time will £3250 amount to £4527. 13s. $1\frac{1}{2}$d. at $4\frac{1}{2}$ per cent. per annum?

6. Find the simple interest on £8450 for 5 years at 3½ per cent. per annum.

7. In how many years will £1230 amount to £2152. 10s. at 7½ per cent.?

8. Find the simple interest on £3235 for 4 years at 3¼ per cent.

9. In what time will £425 amount to £652. 18s. 1½d. at 6½ per cent. per annum?

10. Find the simple interest on £7225 for 4 years at 7½ per cent.

11. At what rate per cent., simple interest, will £1881 amount to £2539. 7s. in 7 years?

Compound Interest.—When the interest is added to the principal so that at the beginning of each year the principal is greater than at the commencement of the previous year, the principal is said to increase at compound interest; compound interest is thus interest upon interest.

Ex. 1. Find the compound interest on £500 for 4 years at 3 per cent.

Since £100 becomes £103 at the end of the first year.

Then £1	,,	$£\frac{103}{100}$	,,	,,
or £1	,,	1·03	,,	,,
∴ £500	,,	500(1·03)	,,	,,

In a similar manner the amount at the end of the second year is 1·03 times the amount of the principal for the second year.

$$\therefore \text{ amount at the end of the second year} = 500 \times (1\cdot03)^2.$$

$$\text{Hence amount at the end of the fourth year} = 500 \times (1\cdot03)^4.$$

$$\therefore \text{ compound interest} = 500(1\cdot03)^4 - 500.$$

This may be put in a more general form, as follows:

If A denote the amount which a given principal P becomes at the rate of r per cent. for n years, then

$$A = P\left(1 + \frac{r}{100}\right)^n.$$

Thus, in the preceding example,

$$P = 500, \ r = 3, \ n = 4;$$

$$\therefore \ A = 500\left(1 + \frac{3}{100}\right)^4 = 500(1\cdot03)^4.$$

Hence, by logarithms,

$$\begin{aligned} \log A &= \log 500 + 4\log 1{\cdot}03 \\ &= 2{\cdot}6990 + {\cdot}0512 \\ &= 2{\cdot}7502; \\ \text{corresponding antilog} &= 5626; \\ \therefore\ A &= £562{\cdot}6. \end{aligned}$$

The fraction ·6 can be converted into shillings and pence if necessary, as on p. 55.

EXERCISES. LIII.

1. Find the compound interest on £2000 in 3 years at 4 per cent.

2. Find the difference between the simple and compound interest of £635. 12s. for 4 years at 3 per cent.

3. Find the amount of 100 guineas in 3 years at 4¼ per cent.

4. Find the amount of £400 in 2½ years at 4 per cent.

5. Find the difference between the simple interest and the compound interest on £2343. 15s. for 2 years at 4 per cent.

6. In how many years will £1000 amount to £1800 at 5 per cent. compound interest?

7. What sum of money put out at compound interest for 2 years at 5 per cent. will amount to £100?

8. Find the amount of £7200 in 3 years at 4 per cent. compound interest.

9. Find the amount of £3650 in 3 years at 5½ per cent. compound interest (neglecting fractions of a penny).

10. Find the amount of £89. 10s. in 3 years at 5 per cent. compound interest.

11. Find the amount in 3 years of £2425 at 4 per cent. compound interest (neglecting fractions of a penny).

12. Find the amount of £2540 in 3 years at 4 per cent. compound interest (neglecting fractions of a penny).

13. Find the compound interest on £133. 6s. 8d. for 3 years at 2½ per cent.

14. What sum will amount to £463. 1s. in 3 years at 5 per cent. compound interest?

Discount.—When a sum of money is paid before it is due, an abatement is made; this abatement is called *discount.*

Thus, for example, the amount of £500 for one year at three per cent. will be £515, hence the *present value* of £515 due at the end of a year is £500; that is, the present value of a given principal due at the end of a given time is that sum, which, if put out to interest, will amount to the sum due.

$\therefore$ sum due = present value + its interest for the given time.

Ex. 1. What is the discount on £286. 13s., payable two years hence, reckoning compound interest at 5 per cent.?

$$£286.\ 13\text{s.} = £286{\cdot}65.$$

$$(\text{Present value}) \times (1{\cdot}05)^2 = 286{\cdot}65\,;$$

$$\therefore\ \text{present value} = \frac{286{\cdot}65}{(1{\cdot}05)^2} = 260\,;$$

$$\therefore\ \text{discount} = £286{\cdot}65 - £260$$

$$= £26.\ 13\text{s.}$$

The above is called *True Discount* to distinguish it from its commonly calculated value, or what is known as *Banker's Discount*, which is greater than the true discount, or, in other words, when a banker discounts at 5 per cent. the interest charged is rather more than 5 per cent.

Ex. 2. If a sum of £1000 becomes due 3 months hence, what is its present value as commonly calculated, and what as correctly calculated, interest being reckoned at 5 per cent.?

Present value of £1000 as commonly calculated:

	£	s.	d.
Sum due =	£1000	0	0
Interest on £1000 for 3 months at 5 per cent,	12	10	0
$\therefore$ present value,	£987	10	0

Present value as correctly calculated:

Amount of £1000 in 3 months at 5 per cent. = £1012. 10s.

$\therefore$ present value of £1012. 10s. is £1000,

$$\text{,,}\quad \text{,,}\quad £1 \quad \text{is } \frac{1000}{1012{\cdot}5},$$

$$\text{,,}\quad \text{,,}\quad £1000 \quad \text{is } \frac{1000 \times 1000}{1012{\cdot}5}$$

$$= £987.\ 13\text{s.}\ 1\text{d.}$$

Hence the difference is 3s. 1d.

EXERCISES. LIV.

1. What is the present worth of a bill for £214. 13s. 4d. due 60 days hence at 5 per cent.?
2. Find the present worth of £1368. 8s. due 8 months hence at $5\frac{1}{2}$ per cent.
3. Find the discount on £399. 16s. 1d. due 81 days hence at 5 per cent.

4. Find the present value of £48. 10s. due two years hence, compound interest being reckoned at 4 per cent.

5. Find the proportion between the interest of £100 for 4 years, at 6¼ per cent., and the discount of the same sum payable at the end of 4 years, at the same rate of interest.

6. Find the present value of a bill for £174. 1s. 10½d. due in 128 days, interest being at the rate of 5 per cent.

7. Find the discount on £663. 17s. due 6 months hence at 4 per cent.

8. What ready money will discharge a debt of £1656. 11s. 10d. due 8 months hence at 4⅞ per cent. per annum?

9. What ready money will discharge a debt of £1056. 18s. due 4 months hence at 4⅞ per cent. simple interest?

10. Find the true discount on £142. 1s. 9d. due 18 months hence at 3½ per cent. per annum.

11. Find the discount on £51. 15s. 10d. due 4⅓ years hence at 3 per cent. simple interest.

Summary.

Arithmetical Progression.—Quantities which increase or diminish by the same amount at each step are said to be in Arithmetical Progression.

The last term is given by $l = a + (n-1)d$.

Sum of n terms, $s = \frac{n}{2}(a + l) = \frac{n}{2}\{2a + (n-1)d\}$.

Geometrical Progression.—In a Geometrical Progression the quantities increase or decrease by a constant factor.

The last term is given by $l = ar^{n-1}$.

The sum of n terms by $s = \frac{a(r^n - 1)}{r - 1} = \frac{a(1 - r^n)}{1 - r}$.

Sum of a series of terms to infinity is $\frac{a}{1-r}$.

The **Arithmetic Mean** of two numbers is one-half the sum of the numbers.

The **Geometric Mean** is the square root of the product of the numbers.

Interest.—Money paid for the loan of a sum borrowed at a fixed rate is called interest, and is called **Simple** Interest when the principal remains unaltered, but **Compound** Interest when the principal continually alters.

Simple Interest. Interest $= \frac{P \times r \times t}{100}$.

Compound Interest. Amount $= P\left(1 + \frac{r}{100}\right)^n$.

Discount.—The abatement made when a sum of money is paid before it is due is called *discount.*

CHAPTER XIV.

ORTHOGRAPHIC PROJECTION. TRUE LENGTHS AND INCLINATIONS OF LINES REFERRED TO TWO AND THREE CO-ORDINATE PLANES.

Lines.—Lines may be straight or curved, or straight in one part of their length and curved in another.

Straight Line.—A straight line may be defined for practical purposes as the shortest distance between two points; or as that line which lies evenly between its extreme points.

Planes.—A plane is a surface such that the straight line joining any two points on it lies wholly in that surface.

Perhaps a clear notion of what this definition implies may be obtained by using a flat sheet of paper, as in Fig. 75. If any two points, *A* and *B*, on the surface of the paper be selected, it will be seen that the line joining them lies in the surface. Next proceed to bend or crease the paper between the points, as by bending along *CD*. The surface no longer remains in one plane, and the shortest, or straight line, joining the two points *A* and *B*, does not lie in the surface.

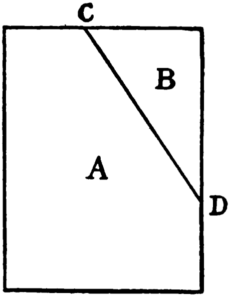


FIG. 75.—Plane surface.

The intersection of two planes is a straight line, because the straight line joining any two points in their line of intersection must lie in both planes.

From a sheet of cartridge paper or thin cardboard cut out two rectangular strips. Cut one narrower than the other, as shown at *A* (Fig. 76). To pass *A* through *B*, without bending or creasing, it is necessary to cut a slit *S* in *B*.

The two planes *A* and *B* are now easily seen to intersect in the straight line *S*.

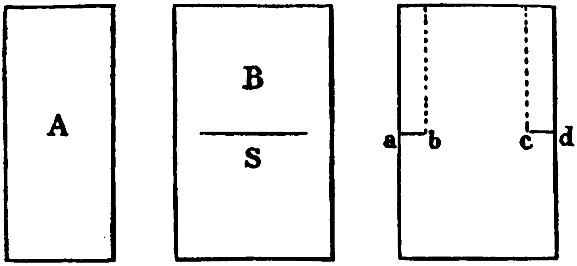


FIG. 76.

Instead of using one strip narrower than the other a useful model of what are called the vertical and horizontal planes can be made by using two equal strips of paper or cardboard, along the centre of one of which a slit is made as shown at *S* (Fig. 76).

In the other strip the two slits, indicated by the lines *ab*, *cd*, are so cut that when folded over the slip can be passed through the slit *S*. When the two pieces are opened out, a model of the vertical and horizontal planes of projection, as shown in Fig. 77, is formed. The two planes are supposed to rotate about the line *xy* as a hinge, until one continuous surface or plane is obtained, corresponding to a sheet of paper.

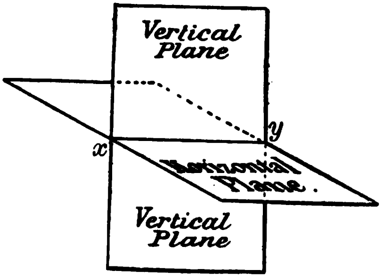


FIG. 77.—Model of the co-ordinate planes of projection.

Descriptive Geometry.—By the methods of descriptive geometry the form of any figure may be determined by constructions drawn to scale. These constructions are made on one plane only, and it is possible by this means to represent a solid having the three dimensions, length, breadth, and thickness, on a plane having but two dimensions, length and breadth, such as a sheet of paper.

This is effected by what are called projections.

Projections of a Point.—The position of a point in space is determined when its projections on two intersecting planes are given.

Thus, if a point *A* (Fig. 78) be given, the *projection* **a** *of the point* on the horizontal plane is the intersection of a perpendicular let fall from the point on to that plane. In a similar manner, if a perpendicular be drawn from *A* to the vertical plane the intersection of the perpendicular with this plane is the *projection of the point* **a′** on the vertical plane.

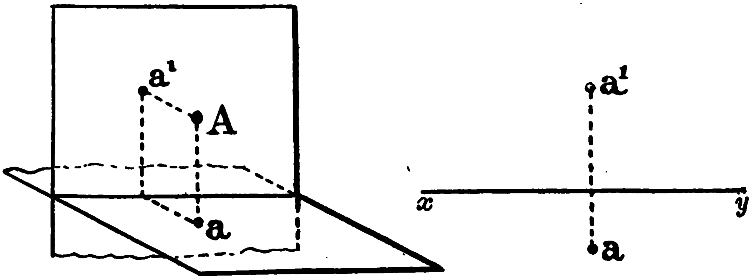


Fig. 78.—Projection of a point.

The two *planes of projection*, intersecting in a line *xy*, when in position are usually at right angles to each other and are always referred to as the *horizontal plane of projection*, or shortly, H.P., and the *vertical plane of projection*, or V.P. The point **a** on the H.P. is called the *plan* of the point, and the point **a′** on the V.P. the *elevation* of the point *A*.

These two projections determine definitely the position of the point *A* in space, because if from **a** and **a′** perpendiculars are drawn to the H.P. and V.P. respectively, the point *A* will lie at some point in the first, and also at some point in the second perpendicular, and therefore must be at the point of their intersection.

It is essential to note that the given point and its projections must lie in one plane. Hence, if the perpendiculars do not intersect, the two projections *a* and *a′* are not the projections of the same point.

The distance *Aa* is equal to the distance of *a* from *xy* and the given point from the H.P.; and *Aa′* is equal to the distance of *a′* from *xy* and the point from the V.P.

Thus, the distance of *the plan of a point* from the ground line (*xy*) indicates *the distance of the point from the* V.P.; and the distance of the *elevation of a point* from the ground line indicates *the distance of the point from the* H.P.

Other Planes of Projection.—As other planes of projection besides those named may be used, the elevation of a point is marked by an italic letter with a dash at the upper right-hand corner of the letter, thus *a′*. The plan of a point is marked by the same italic letter, but without a dash, as *a*.

The capital letter *A* is only used to specify the point itself, and is not shown except when a perspective view of the planes is made, as in Fig. 78.

Ex. 1. A point *A* is ·5″ in front of the V.P. and ·7″ above the H.P. Another point *B* is at a distance of ·6″ from the H.P. and ·4″ from the V.P. Show the projections of the two points.

Commence by drawing the ground line *xy* (Fig. 78); draw a line *aa′* perpendicular to *xy*.

Make the distance from *xy* to *a′* equal to ·7″ and from *xy* to *a* equal to ·5″ as shown.

Then *a′*, *a* are the required projections of the point *A*.

In a similar manner measuring distances ·6″ above and ·4″ below *xy*, on a line *b′b*, drawn perpendicular to *xy*, the two projections *b′b* of a point *B* can be obtained.

Projections of a Line.—The projection of a line *AB* on a plane *MN* (Fig. 79) is obtained as follows:

From *A* and *B* let fall perpendiculars (as shown by the dotted lines), on the plane *MN*. The line joining the points where these dotted lines meet the plane is the projection required.

Angle between a line and plane, *or the inclination of a line to a plane*, is the *angle between the line and its projection on the plane*; thus, if *BA* produced meets the plane as shown (Fig. 79), the inclination of the line to the plane is the angle between the line and its projection on the plane.

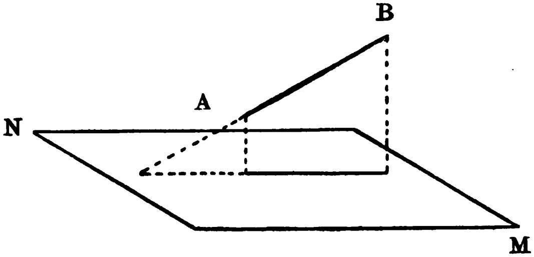


FIG. 79.—Angle between a line and a plane.

A line may be in any one of the following positions:

(1) *Parallel to both planes of projection.*

(2) *Perpendicular to either plane of projection.*

(3) *Parallel to one plane of projection and inclined to the other.*

(4) *Inclined to both planes of projection.*

(1) If a *line is parallel to both planes of projection*, the projections of the line will be parallel to the line of intersection of the planes. Thus, in Fig 80 the line AB is parallel to xy. The plan ab and the elevation $a'b'$ are also both parallel to xy.

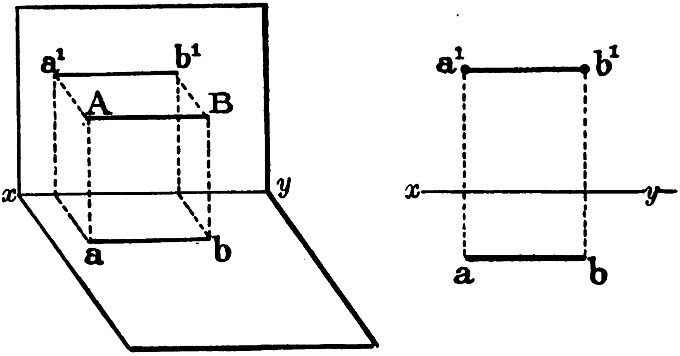


FIG. 80.—Projections of a line parallel to both planes of projection.

(2) *Line perpendicular to one plane and parallel to the other.* In Fig 81 the line CD is perpendicular to the V.P. and parallel

to the H.P. The plan is a line *cd* perpendicular to *xy*. The elevation is a point *c'd'*.

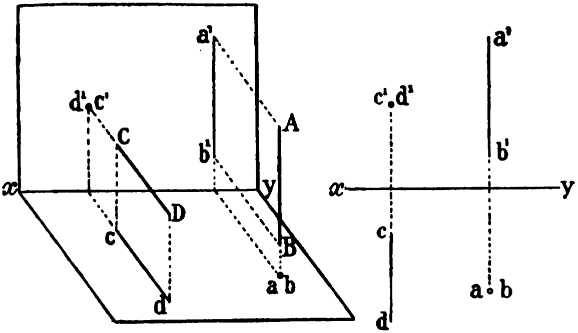


FIG. 81.—Projections of a line, perpendicular to one plane, and parallel to the other.

In a similar manner, when a line is perpendicular to the H.P., the plan of the line is a point and the elevation a line as shown in Fig. 81.

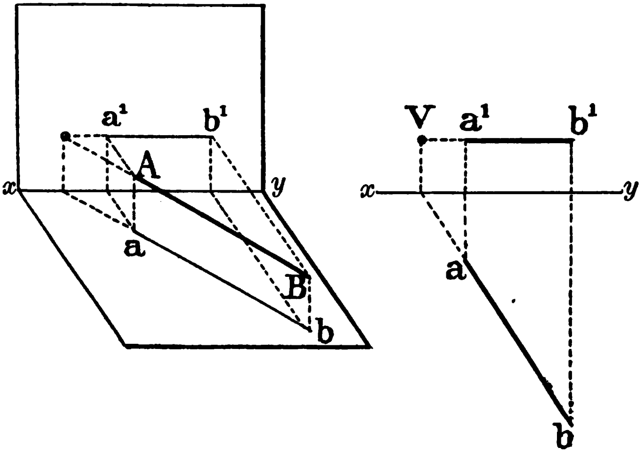


FIG. 82.—Projections of a line parallel to H.P. inclined to V.P.

By using a piece of paper bent at right angles about a line *xy* and a pencil (or a piece of wire) the student should verify every one of the constructions shown. When this is done the ordinary or orthographic projections, as shown on p. 205, will not present any difficulty.

(3) *Parallel to one plane and inclined to another.* The line *AB* (Fig. 82) is shown to be parallel to the H.P. and inclined to the V.P. Hence, the elevation of the line is a line parallel to *xy* and the plan a line inclined to *xy* as shown.

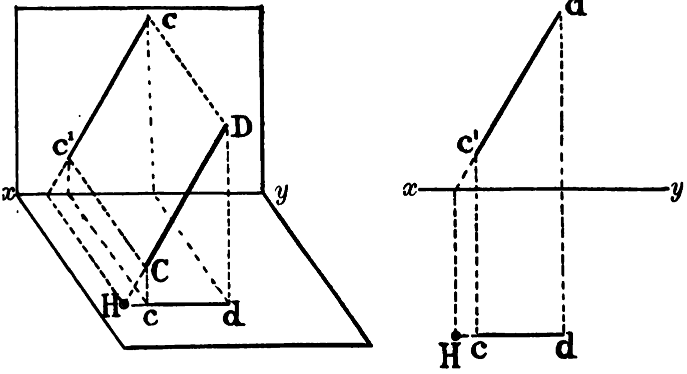


FIG. 83.—Projections of a line, parallel to V.P., inclined to the H.P.

When, as in Fig. 83, the line is parallel to the V.P. but inclined to the H.P., the projections of the line are, as in the last case, a line inclined to and a line parallel to *xy*, in this case the plan of the line is parallel, and the elevation inclined to *xy*.

True Length of a Line.—It is obvious by using a piece of straight wire or a pencil laid on a sheet of paper on which the length of wire or pencil has been marked, that, so long as the pencil remains in contact with, or is kept parallel to, the plane, and at any given distance from it, the projection of the line on the plane to which it is parallel will be its true length.

When a line is inclined to a plane its projection on that plane is shorter than its real length.

As the inclination of the line to the plane is increased the projection of the line gets shorter and shorter until when the line becomes perpendicular to the plane, the projection is a point, as shown in Fig. 81.

In Figs. 82 and 83, which represent the ordinary projections of the lines *AB* and *CD*, it will be seen that the true length of the line *AB* is the plan *ab* and the true length of *CD* the elevation *c'd'*.

In the first case the line is parallel to the H.P., and in the second case the line is parallel to the V.P.

(4) We have found that when a line is inclined to one plane and parallel to the other its projection on the plane to which the line is parallel will equal its true length; and the projection of a line on a plane to which the line is not parallel is *shorter* than the real length.

It follows therefore that when a line is *inclined to both planes* of projection the two projections of the line (plan and elevation) are shorter than the real length of the line.

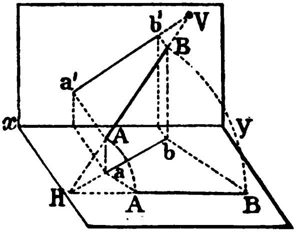


Fig. 84.—Line inclined to both planes of projection.

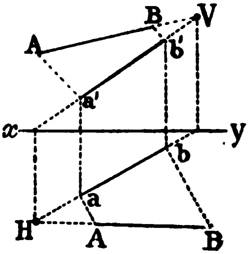


Fig. 85.—Projections of a line inclined to both planes of projection.

In Fig 84 a line *AB* and its projections *a'b'*, *ab*, are shown, and in Fig. 85 the ordinary or orthographic projections of the same line are given. As both projections of the line are shorter than the real length, we require to determine the true length of the line and also its inclinations to both planes of projection.

To obtain the true length and true inclinations of a line to both planes of projection.—Referring to Fig. 84 it will be obvious that the line *AB*, its projectors, *Bb* and *Aa*, and projection *ab*, form a four-sided figure *ABba*. If we suppose this four-sided figure to rotate about *ab* as axis, the points *A* and *B* will describe arcs of vertical circles the plans of which are the lines *aA*, *bB*. *A* and *B* are the points where the extremities of the revolving line meets the H.P.; *aA* and *bB* are in each case lines drawn at right angles to the projection *ab*. The line *AB* on the H.P. is the true length of *AB* and the inclination of *AB* to *ab* is the inclination of *AB* to the H.P.

This inclination to the H.P. may be obtained by drawing from *A* a line parallel to *ab* or by producing *BA* and *ba* to meet at *H*. The method just referred to of rotating a figure into coincidence with a plane is called *rabatment*, and *AB* is the line rabatted on to the H.P.

In a similar manner, the line could be rabatted into the V.P., and the true length also the inclination ϕ to the V.P., as before obtained.

Inclination of a Line to a Plane.—The angle which the line *CD* or *CD* produced makes with its projection *cd* is the inclination of the line to the H.P. Thus producing the line *DC* (Fig. 83) to meet the H.P. at *H*, the projection *dc* will if produced also pass through *H*, and the inclination of the line to the H.P. is the angle *DHd*.

The angle so determined could be measured with a protractor and its magnitude written down. It is usual, however, to indicate the angle by the Greek letter θ.

In a similar manner, the inclination of the line *AB* (Fig. 84) to the vertical plane is the angle *BVb′*. As before, its magnitude could be obtained by means of a protractor, but it is conveniently denoted by the Greek letter ϕ.

Traces of a Line.—When a line is inclined to a plane it either penetrates the plane or would do so if produced.

If the line is inclined to the H.P., its point of intersection with that plane is called the *horizontal trace*. If inclined to the vertical plane its point of intersection with the V.P. is called the *vertical trace* of the line.

When a line is parallel to a plane it does not penetrate the plane even though produced, and therefore has no trace on the plane.

In Fig. 83 the point *H* is the horizontal trace of the line *CD*, and as the line is parallel to the V.P. the line has no vertical trace.

Similarly, the point V (Fig. 82) is the vertical trace of the line AB.

Ex. 1. *Given the projections* $a'b'$, ab *of a line* AB (Fig. 85) *to find*

(1) *The true length of the line.*

(2) *The true inclinations to both planes of projection.*

(3) *The traces of the line.*

(1) Draw bB, aA at right angles to ab, and make bB equal to the height of B above the H.P.; also aA equal to the height of A. Join AB, then AB is the true length required.

(2) If BA and ba be produced to meet at H, the angle BHb is the inclination θ of the line to the H.P.; or draw through A a line parallel to ab.

To obtain the inclination of the line to the V.P., draw $b'B$ and $a'A$ perpendicular to $a'b'$, the distances $b'B$ and $a'A$ are respectively equal to the distances of the two points B and A from the V.P.

Join BA and produce AB and $a'b'$ to meet at V; then AB is the true length of the line, and the inclination ϕ of the line to the V.P. is the angle AVa'.

(3) The two points H and V are the traces H and V of the line, the former the horizontal, and the latter the vertical trace of the line.

A simple model can be easily made as follows from cartridge drawing-paper or thin cardboard:

Draw the projections $a'b'$, ab of a line AB (Fig. 85). Rabat as shown the line into the H.P. and V.P.

Cut completely through the paper along the lines aA, $a'A$; bB, $b'B$, AB.

The four-sided figure $aABb$ may be rotated about ab as hinge until it is at right angles to the H.P.; in a similar manner the four-sided figure $a'ABb'$ can be rotated about $a'b'$ until the plane of the figure is perpendicular to the V.P. The two lines, each denoted by AB on the H.P. and V.P. respectively, will then coincide, and the rabatment into the H.P. or V.P. is clearly seen to give the required true length. In this manner a simple and effective model may easily be made. To enable the cardboard to bend readily on the two lines referred to, it will be found advantageous to cut the cardboard half through on the reverse side of the diagram.

Ex. 2. Given the projections of a line (Fig. 86) to determine its true length and inclination to the H.P.

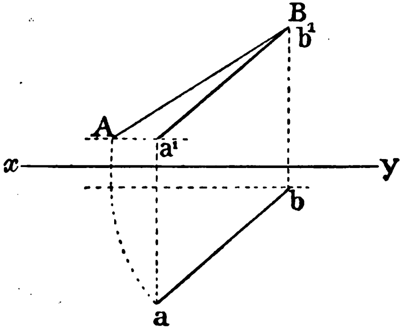


FIG. 86.—Given the projections of a line to determine its true length and inclination to the H.P.

With centre b and ba as radius describe an arc of a circle.

Draw a line parallel to xy, intersecting the circular arc.

A projector drawn from the point of intersection and intersecting a horizontal line through a' gives a point A; by joining this point to b' the true length required is obtained.

The angle $b'Aa'$ is the inclination of the line to the H.P.

EXERCISES. LV.

Show in plan and elevation:

1. (*a*) A point 2″ from the ground line and 1¼″ from the vertical plane of projection.

(*b*) A line parallel to and 1¼″ from the vertical plane of projection, and inclined at 40° to the horizontal plane.

N.B.—Elevation and plan are to be properly distinguished in each case.

2. A is a point in the vertical plane 1¼″ above the horizontal plane; B is a point in the horizontal plane 1¾″ from the vertical plane. The real distance from A to B is 3″. Draw the plan and elevation of the line joining A and B.

3. A point 1·5″ from both planes of projection is distant 3·25″ from another point 2·25″ from both planes of projection. Obtain the projections of the two points.

4. A point is 2″ from the vertical and 1·75″ from the horizontal plane. Determine a point on the ground line, distant 3·25″ from this point.

5. c' is the elevation of a point on the given line $a'b'$, ab; d is the plan of another point on the same line (Fig. 87). Determine the real length of the line CD.

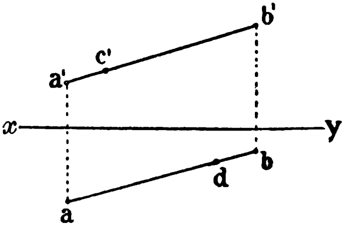


FIG. 87.

6. Construct an equilateral triangle of $2\frac{1}{8}''$ side. Consider the triangle to be the plan of a triangle *A*, *B*, *C*, such that the heights of the angles *A*, *B*, *C* above the horizontal plane are $\frac{7}{8}''$, $1\frac{5}{8}''$, and $1\frac{1}{4}''$ respectively. Draw the plan of a horizontal line lying in the plane of the triangle and passing through *C*.

7. A horizontal line 1·25″ above the horizontal plane is terminated by the two given lines *ab*, *a′b′*; *cd*, *c′d′* (Fig. 88). Determine the real length of this line.

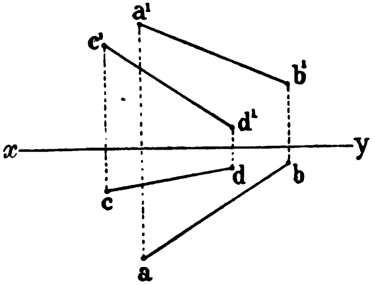


FIG. 88.

8. *V* and *H* are the vertical and horizontal traces of a line; H_1 is the horizontal trace of a second line, which bisects the first (Fig. 89). Draw the projections of the second line.

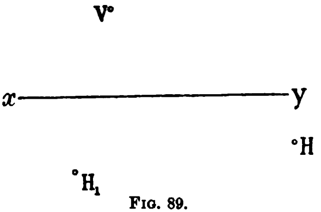


FIG. 89.

9. *q*, *q′* are the projections of a point on a line whose elevation is *p′q′*. Find the plan (*pq*), the line being inclined at 30° to the horizontal plane of projection.

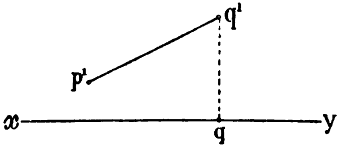


FIG. 90.

Given the true length of a line and its inclinations to both planes of projection, to obtain its projections.

This is the converse of the problem on page 207, and the method adopted can be illustrated by the following example:

Ex. 1. The true length of a line is 3″, its inclinations to the H.P. and V.P. 50° and 20° respectively. Obtain the projections of the line.

If a line is inclined at any given angle to a plane, the *length* of its projection on that plane can easily be determined; the *position* of the projection will depend upon the given data.

The *length* of the projection of a line 3″ long and inclined at 50° to H.P. may be obtained by drawing the line Ab' of the required length and at the given inclination as shown in Fig. 91.

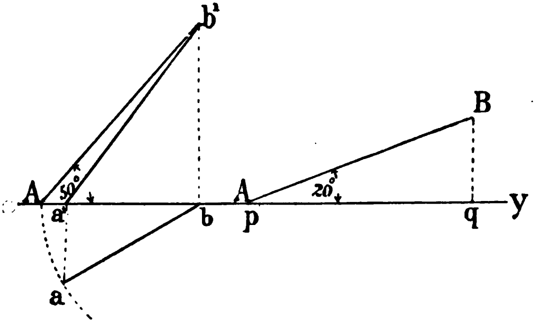


FIG. 91.

If from b' a projector $b'b$ is drawn, the line Ab is the *plan length* of the line.

With b as centre, radius bA, describe an arc of a circle; the plan of the line will be some radius (such as ba) of this arc.

The projection of the line on a plane to which it is inclined at 20° is shown at pq (Fig. 91), hence pq is the length of the elevation.

With b' as centre, radius pq, describe an arc cutting xy at a'; draw $a'a$ at right angles to xy, intersecting the circular arc at a, this gives the plan of point a'. Join a to b and a' to b'.

The lines $a'b'$, ab are the required projections.

The construction shown can be put in a more convenient form as follows:

As in the last example draw the line Ab' at 50° and 3″ long; a projector from b' gives the plan length Ab.

To determine the elevation of the line, draw $b'q'$ making an angle of 20° with Ab' (Fig. 92). Make the angle $b'q'A$ a right angle; then $b'q'$ is the length of the elevation.

With *b'* as centre and radius *b'q'* describe an arc cutting *xy* at *a'*.

Join *a'* to *b'*, then *a'b'* is the elevation required.

To obtain the plan project *a'* to meet an arc of a circle drawn with *b* as centre and *bA* as radius at *a*; join *a* to *b*.

The projections of the line are *a'b'*, *ab*.

The angle at *q'* is a right angle and this is readily made by drawing on *Ab'* as diameter a semicircle as shown.

The point of intersection *q'* of the line *b'q'* with this semicircle gives the length of elevation required.

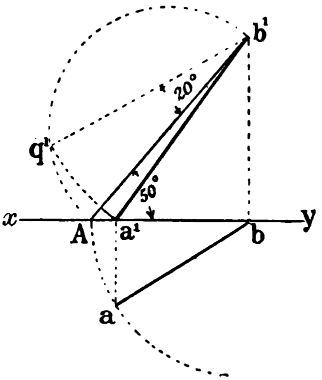


FIG. 92.—To obtain the projections of a line given its true length and inclinations.

Indexed Points, or Plans.—The elevation of a point gives the distance of the point from the horizontal plane, and if the plan of a point and this distance be given an elevation can at once be drawn.

If, as shown in Fig. 93, the indexed plan of a point *a* is given, the index consisting of a number (which is placed near the bottom of the letter and on the right-hand side of it) denoting that the height of the point above the H.P. is 6 units. The elevation *a'* can at once be drawn by drawing a ground line *xy* and making the distance of the elevation *a'* from it equal to 6 units.

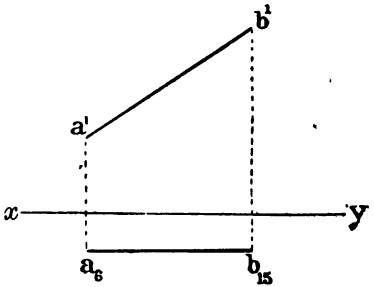


FIG. 93.—Indexed points.

In a similar manner the elevation of a point *b*, the indexed plan of which is given in Fig. 93, is obtained by making the distance from $xy = 15$ units.

It will be obvious that by joining a to b we obtain the indexed plan of a line, the true length and elevation of which is $a'b'$.

The horizontal trace of the line is obtained as before by producing $b'a'$ to meet xy, and from the point of intersection drawing a perpendicular to meet the plan ba produced at H.

Then H, as in Fig. 93, is the horizontal trace required.

As the line is parallel to the V.P. the inclination of the line to the H.P. is the angle θ which the line $b'a'$ produced makes with xy.

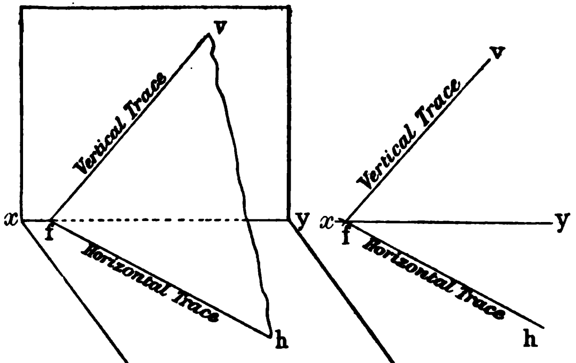
EXERCISES. LVI.

1. An equilateral triangle of 2″ side has its angular points indexed 2, 5, and 9. Determine the true form of the triangle.

2. The plans of three points form a right-angled triangle $a_5b_{15}c_{25}$. The two sides containing the right angle are 2″ and 3″. Find true shape.

3. Draw the projections of a line 3″ long inclined at 33° to the V.P. and 53° to H.P., (a) when the line passes through xy, (b) when it does not pass through xy.

4. Draw the projections of a line 3½″ long inclined at 35° to the V.P. and 45° to the H.P.



(I) (II)

Fig. 94.—Oblique Plane.

Planes.—As already indicated, the intersection of two planes is a straight line. This line is called the *trace* of one plane on the other. Thus the line *xy* (ground line) is the intersection of the horizontal and vertical planes of projection.

In like manner, the intersection of any other plane, such as *Vfh*, Fig. 94, (i.) with either of the co-ordinate planes is a straight line. The intersection *fh* with the H.P. is the horizontal trace of the plane, and the intersection *fV* with the vertical plane is the vertical trace; a plane is usually and correctly represented by its traces *Vf*, and *fh*, Fig. 94 (ii.). The two traces are denoted by the letters H.T. and V.T.

Oblique Plane.—A plane which is inclined to both planes of projection is called an oblique plane. Using a model of the co-ordinate planes and pieces of cardboard or thick paper cut to suitable shapes, not only the plane just described, but, in addition, other forms and positions may be represented. Of these the two shown in Figs. 95 and 96 are perhaps the most important.

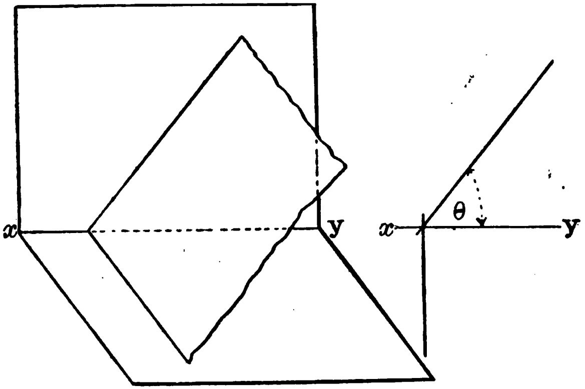


Fig. 95.—Inclined Plane.

Inclined Plane.—As will be seen from Fig. 95, the plane is perpendicular to the V.P. and inclined to the H.P. Hence the H.T. of the plane is perpendicular to *xy*, and the V.T. makes an angle with it. The angle which the vertical trace, V.T., of the plane makes with *xy* is the inclination of the plane to the H.P.

The magnitude may be indicated in the usual way in degrees, etc., and is usually denoted by the letter θ.

Vertical Plane.—In Fig. 96 the plane shown is perpendicular to the H.P. and inclined to the V.P. Hence the V.T. of the plane is at right angles to xy, and the H.T. inclined at an angle to it, which is usually denoted by ϕ.

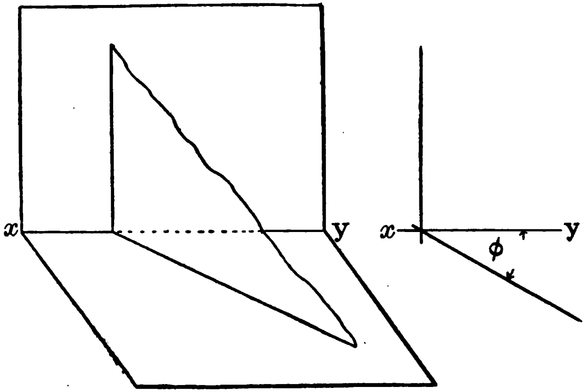


FIG. 96.—Vertical Plane.

In the two cases just referred to, the angles made by the traces of the plane with xy indicate in each case the true inclinations of the planes to the planes of projection. This is not the case with the oblique plane shown in Fig. 94. The inclinations of the plane to both planes of projection require to be obtained by construction; this may be carried out as follows:

Imagine the plane to rest on a cone, the base of the cone in the H.P., its axis in the V.P., and its apex on the V.T. Then the angle made by the slant side of the cone will obviously represent the inclination of the plane.

In a similar manner the inclination to the V.P. is obtained, the plane touching a cone whose axis is in the H.P. and whose base is in the V.P. The construction is as follows:

At any point V in xy (Fig. 97) draw $V'Vh$ perpendicular to xy, and Vn perpendicular to fh, meeting fh at n. With V as centre, Vn as radius, describe an arc of a circle, intersecting xy at a'.

Join a' to V', then $V'a'V$ is the inclination of the plane to the H.P., usually denoted by the letter θ.

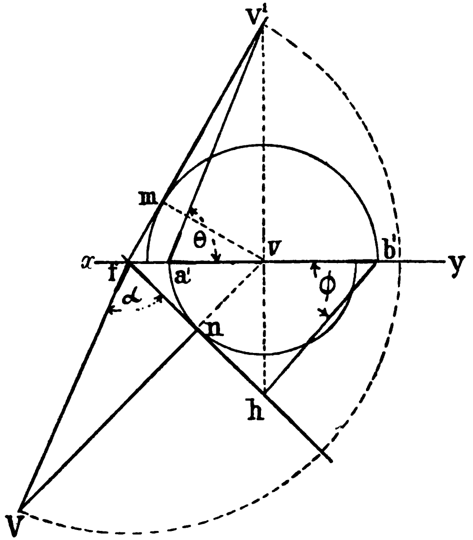


Fig. 97.—Inclinations: Oblique Plane.

To obtain the inclination of the plane to the V.P. a similar construction may be used, *i.e.* draw Vm perpendicular to the V.T.; With V as centre, radius Vm, draw arc of circle as shown, intersecting xy at b', join b' to h; $Vb'h$ is the inclination of the plane to the V.P., usually denoted by the letter ϕ.

Angle between the Traces.—The angle between the traces may be found by obtaining the true shape of the triangle $V'fn$. Actually, of course, VV' is perpendicular to Vn, so that $a'V'$ is the true length of a line joining V' and n. Hence draw a triangle having its adjacent sides equal to nf and fV' respectively, and its base equal to $a'V'$. Then the angle a at f will be that between the two traces.

Conversely, given that a plane is inclined at θ degrees to the horizontal and ϕ degrees to the vertical plane, to draw its traces.

If two cones envelope a common sphere, their bases in the planes of projection, and their slant sides inclined at the given angles, then the plane which touches these cones will have the required inclination. The construction is as follows:

At any point A in xy (Fig. 98), make the angle $VAM = \theta$. Draw VH perpendicular to xy and MN at right angles to VA. With M as centre, radius MN, describe a circle. Draw BH touching this circle and making an angle with xy equal to ϕ degrees.

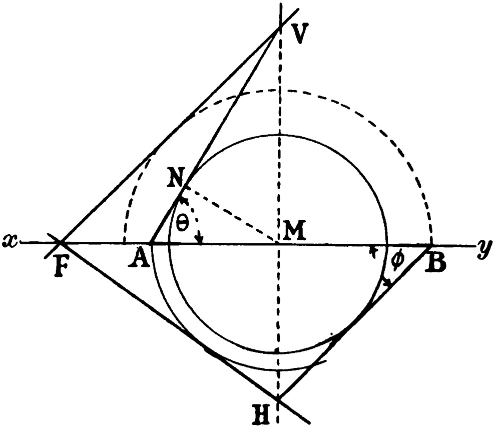


FIG. 98.—Given inclination of a plane to obtain its traces.

With M as centre, radius MA, describe an arc of a circle as shown. The H.T. of the plane required will touch this arc and will pass through the point H. The point of intersection F of the line so drawn with xy is one point in the vertical trace. Join F to V, then VFH are the traces required.

Another Method.—Using the construction shown on p. 213, draw the projections of a line inclined at $(90 - \theta)$ degrees to the H.P. and $(90 - \phi)$ degrees to the V.P., and draw the traces of the required plane respectively perpendicular to the projections (plan and elevation) of the line.

Note.—The sum of the given inclinations θ and ϕ must lie between 90° and 180°.

EXERCISES. LVII.

1. The horizontal and vertical traces of a plane make respectively angles of 55° and 30° with the ground line. Determine the inclinations of the plane to both planes of projection; also the angle between the traces.

2. The H.T. of a plane makes 45° with xy. Find its V.T. when (i) the plane is at 60° to the H.P.; (ii) the plane is at 45° to the V.P.; (iii) the angle between the traces is 60°.

3. The traces of a plane both make 45° with xy. Find the inclinations of the plane to both planes of projection; also the angle between the traces.

4. Determine the projections of a line inclined at 30° to the horizontal plane, and 40° to the vertical plane. Find the angle which this line makes with a plane which is perpendicular to xy.

5. The traces of a plane make respectively angles of 30° and 60° with xy. Determine and write down the angles which the plane makes with the planes of projection, and with xy.

6. The horizontal trace of a plane makes an angle of 40° with xy; the angle between the traces is 70°. Draw the traces of the plane and find the inclinations.

Three Co-ordinate Planes of Projections.—Another method of representing a point or points in space is by means of the projections on three intersecting, or co-ordinate planes, as they are called; these projections determine the distances of the point from the three planes, and hence, the position of the point is

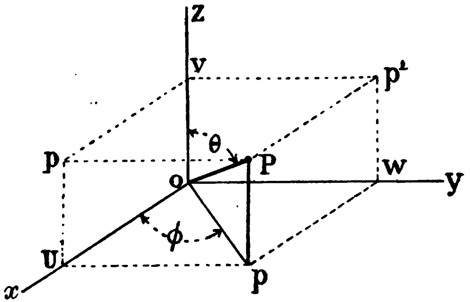


Fig. 99.

determined. The planes are usually mutually at right angles to each other, such as the corner of the cube, or roughly, the corner of a room.

If Fig. 99 denotes one corner of a room, then a portion of the floor may be represented by the horizontal floor or plane sometimes spoken of as the plane xy, one vertical wall by the plane xz, and the other vertical wall at right angles to xz by zy.

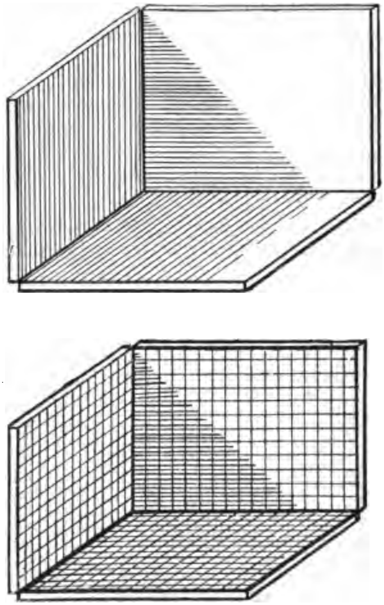


Fig. 100.—Models of the three co-ordinate planes of projection.

It is well to make a model to illustrate the above.

This may consist of a piece of flat board Fig. 100 and two other pieces mutually at right angles to each other.

It will be advisable to have the latter two boards hinged. This enables the two sides to be rotated until all three planes lie in one plane. If the three pieces of wood are painted black they may be ruled into squares as shown, or squared paper may be pasted on them. By means of pieces of straight wire (hat pins answer very well) with one end pointed and the other end terminating in a bead or ball, many problems can be effectively illustrated.

If preferred, a model can be easily made from drawing paper or cardboard. Draw two lines oy and oz on a piece of cardboard at right angles to each other, and cut the cardboard half through, so that it will bend along the dotted lines as shown in Fig. 101.

Draw *oa* at 45° and fold the paper back, so that the edge *oz′* will fall on the line *oz*.

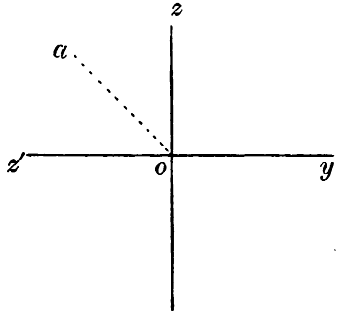


FIG. 101.

We can now fix the position of a point in space by the distance from three such planes. Thus, imagine a point *P* in space; from *P* let fall a perpendicular on the horizontal plane and meeting it in *p* (Fig. 99). *pP* is the distance of the point *P* from the plane *xy* or is the *z* *coordinate* of *P*.

A perpendicular let fall on the plane *yz*, meeting it in *p′*, will give the distance from the plane *yz* or the *x* co-ordinate of the point.

In a similar manner the distance *Pp″* the *y* co-ordinate of the point is the distance of the point from the plane *zx*.

It is obvious that the three projections of a point on three intersecting planes definitely determine the distance of a point from these planes.

The perpendiculars drawn from these three points would intersect at the point itself. Thus the position of the point is completely determined.

In the example we have assumed the *z* co-ordinate to be **above** the plane of *xy*, but the method applies equally to distances **below** the plane.

Hence, a point or object above or below the earth's surface could be specified if two intersecting vertical planes, such as two walls meeting at right angles, were to be found in the neighbourhood. The distances from the two walls, together with the remaining or *z* co-ordinate would completely define the position of the point.

Stories of buried treasure may in this manner be set out. A person unable to carry away treasure might select a place in which two convenient intersecting walls are to be found in the neighbourhood. If deposited at some depth below the surface the treasure could be recovered at any future time provided that the respective distances from the two walls and the depth below the surface were known.

Some of the data we have assumed may perhaps be better expressed by the terms **latitude** and **longitude** of a place on the earth's surface. Thus, at regular distances from the two poles a series of parallel circles are drawn (Fig. 102) and are called *Parallels of Latitude.* The parallel of latitude midway between the poles is called the **Equator.** These parallels are crossed by circles passing through the poles, and are called **meridians of longitude.** Selecting one meridian as a standard (the meridian passing through Greenwich), the position of any object on the earth's surface can be accurately determined. This information, together with the depth below the surface or the height above it, determines any point or place. We may now proceed to work out a few simple examples on this method of projection.

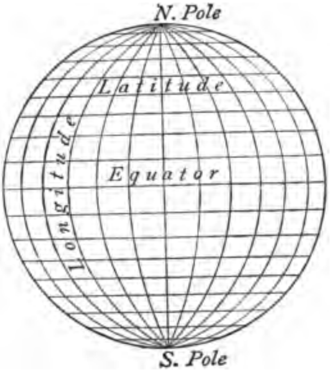


FIG. 102.

Ex. Given the x, y, and z co-ordinates of a point P as $1\frac{1}{2}''$, $2''$, and $1''$ respectively, draw the three projections of the point P on the planes of XY, YZ, and ZX, and measure in each case the length of the projections of the line OP.

Draw the two axes XZ and OY, intersecting at O, and letter as shown in Fig. 103.

The respective co-ordinates are set off along the lines to which they refer.

Thus, the Y co-ordinate of the point is the distance $2''$.

The X co-ordinate of the point is the distance $1\frac{1}{2}''$.

The Z co-ordinate of the point is $1''$.

Join point O to p, p_1, and p_2.

Then Op_1, Op, and Op_2 are the required projections, and these when measured will give the lengths of the projections as follows:

$$Op_1 = 2{\cdot}24'',$$
$$Op = 2{\cdot}5'',$$
$$Op_2 = 1{\cdot}8''.$$

To obtain the true distance of P from the origin O, it is necessary to obtain the true length of a line of which Op is the projection. The true length is obtained by rabatting the line on to the plane. To do this, draw pP perpendicular to Op, and equal in length to the height of P above the plane of XY, this is the distance ap.

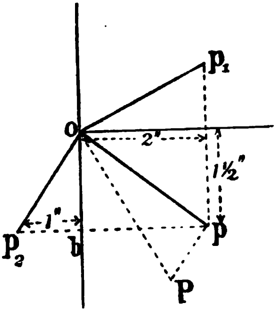


FIG. 103.

Join O to P; then OP is the true distance required $= 2{\cdot}55''$.

EXERCISES. LVIII.

1. The co-ordinates of two points A and B are as follows:

	x	y	z
A	1·72″	·93″	·45″
B	·66″	2·92″	1·37″

Determine and measure the true length of AB, and the angles which the line makes with the planes of reference.

2. The x, y, z co-ordinates of a point A are respectively 2·5″, 3·5″, and 1·5″. Draw the three projections of the line OA where O is the origin. Determine and measure the angles which OA makes with the axes of X, Y, and Z.

3. The perpendicular distances of a point P from the floor and two adjacent sides of a room are respectively 6·4, 3·2, and 5·7 feet, the corresponding distances of a second point Q being 4·0, 6·9, and 2·5 feet. Determine graphically and measure the distance between the two points.

4. Determine graphically the angle subtended at the centre of the earth by two places on its surface, the latitudes of which are respectively 20° North and 60° North, the difference of the longitudes of the places being 50°.

Calculation.—By the methods already described, the results can be obtained to a fair degree of accuracy by graphic construction, but more accurately by calculation, and one method may be used to check the other. In Fig. 99 let θ denote the angle made by OP with the axis of z, and ϕ the angle which the projection Op makes with the axis of x; then we have:

$$x = OU = Op \cos\phi\,; \text{ but } Op = OP \sin\theta\,;$$

$$\therefore\ x = OP \sin\theta \cos\phi\,; \tag{i}$$

$$y = OW = Op \cos pOW = Op \sin\phi\,;$$

$$\therefore\ y = OP \sin\theta \sin\phi, \tag{ii}$$

and

$$z = OP \cos\theta. \tag{iii}$$

Ex. 1. Let $OP = 100$, $\theta = 25°$, $\phi = 70°$.

Then

$$\begin{aligned} x &= 100 \sin 25° \cos 70° \\ &= 100 \times \cdot 4226 \times \cdot 3420 = 14{\cdot}45\,; \\ y &= 100 \sin 25° \sin 70° \\ &= 100 \times \cdot 4226 \times \cdot 9397 = 39{\cdot}71\,; \\ z &= 100 \cos 25° = 100 \times \cdot 9063 = 90{\cdot}63. \end{aligned}$$

Ex. 2. Given the coordinates of a point $x=2$, $y=4$, $z=5$; find OP or r, θ, and ϕ.

$$OP^2 = r^2 = Op^2 + pP^2\,; \text{ also } Op^2 = OW^2 + Wp^2\,;$$

$$\therefore\ r^2 = Wp^2 + OW^2 + pP^2 = x^2 + y^2 + z^2$$

$$= 9 + 16 + 25 = 50\,; \quad \therefore\ r = \sqrt{50} = 7{\cdot}071.$$

From (iii) $\quad z = r\cos\theta = 7{\cdot}071 \cos\theta\,;$

$$\therefore\ \cos\theta = \frac{5}{7{\cdot}071} = \cdot 7071\,; \quad \therefore\ \theta = 45°.$$

From (ii) $\quad 3 = r \sin\theta \cos\phi,$

or

$$\cos\phi = \frac{3}{r\sin\theta} = \frac{3}{7{\cdot}071 \times \cdot 7071}\,; \quad \therefore\ \phi = 53°\ 6'.$$

Direction-Cosines.—As before, denoting OP by r (Fig. 99), then the ratios $\frac{x}{r}$, $\frac{y}{r}$, and $\frac{z}{r}$, are called the direction-cosines of the line OP, and are denoted by l, m, and η respectively. The equation of a line joining the point xyz to the origin is

$$r = \frac{x}{l} = \frac{y}{m} = \frac{z}{n} \tag{iv}$$

The equation of a line passing through the two points (xyz), (abc) is $\frac{x-a}{l}=\frac{y-b}{m}=\frac{z-c}{n}$.

This reduces to (iv) when the second point is the origin, *i.e.* when $a=0$, $b=0$, $c=0$.

Ex. 3. If $x=3$, $y=4$, $z=5$, find r, l, m, and n.

We have
$$r^2=x^2+y^2+z^2=9+16+25=50;$$
$$\therefore\ r=\sqrt{50}=7{\cdot}071;$$
$$l=\frac{x}{r}=\frac{3}{7{\cdot}071}={\cdot}4242;$$
$$m=\frac{y}{r}=\frac{4}{7{\cdot}071}={\cdot}5657;$$
$$n=\frac{z}{r}=\frac{5}{7{\cdot}071}={\cdot}7071.$$

When the given point or points are in the plane of xy, a resulting simplification occurs. Thus, for example, given $x=3$, $y=4$, point P (Fig. 104) is obtained by marking the point of intersection of the lines $x=3$, $y=4$.

In a similar manner point A (2, 2·66) is obtained, and is shown at A. Join P to A, then PA is the line through the points (3, 4), (2, 2·66) or the line
$$y-2{\cdot}66=\frac{4-2{\cdot}66}{3-2}(x-2),$$
or
$$y-1{\cdot}33x=0.$$

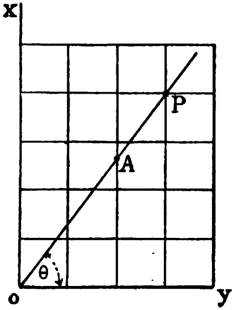


FIG. 104.

Polar Co-ordinates.—If from the point P a line be drawn to the origin, then if the length of OP be denoted by r, and the angle made by OP with the axis of x be θ. When r and θ are known, the position of the point can be determined, and the rectangular co-ordinates can be found.

Conversely, given the x and y of a point, r and θ can be obtained.

Ex. 4. Let $r=20$, $\theta=35°$; find the co-ordinates x and y.

Here
$$x=r\cos 35°=20\times{\cdot}8192=16{\cdot}384;$$
$$y=r\sin 35°=20\times{\cdot}5736=11{\cdot}472.$$

Ex. 5. Given the co-ordinates of a point P (4, 3); find r and θ.

$$r^2=4^2+3^2=25; \quad \therefore \quad r=5;$$
$$\tan\theta=\tfrac{3}{4}={\cdot}75,\ \theta=36°\ 54'.$$

EXERCISES. LIX.

1. If $r=50°$, $\theta=60°$, $\phi=45°$, find x, y, z.

2. A square pillar, side of square 1 foot, bears a load of 40 tons; find the stress per square inch on a cross section and also on a plane section inclined at 25° to the axis.

3. The base of a stone obelisk is a square of 3 feet side and stands above a slate formation inclined at 10° to the horizon; find the stress on a cross section and also on the formation if the load on the obelisk is 200 tons.

4. A straight line makes an angle of 60° with one axis and 45° with another; what angle does it make with the third?

5. The co-ordinates of a point on a line passing through the origin are $x=2$, $y=5$; find θ and r.

6. Given $r=10$, $\theta=25°$; find the co-ordinates x and y.

7. Given $x=3$, $y=4$; find r and θ.

8. Given $x=5$, $y=8$; find r and θ.

9. Given $r=100$, $\theta=15°$; find x and y.

10. Given $r=50$, $\theta=20°$; find x and y.

11. A plane area of 50 square feet is inclined at an angle of 25° to the horizontal; find its horizontal and vertical projections.

12. Given the rectangular co-ordinates of a point A, 3″, 4″, and 2″ respectively; find the distance of the point A from the origin, and the angles made by the line OA with the three axes.

13. The polar co-ordinates of a point A are 3″, 40°, and 50°; find the rectangular co-ordinates. Find the rectangular co-ordinates of the line OA.

Summary.

A **Straight Line** may be defined as *the shortest distance between two points.*

A **Plane** is a surface such that *the straight line joining any two points on it lies wholly on that surface.*

The **Intersection of Two Planes** is *a straight line.*

Projection of a Point.—The position of a point in space is completely determined when its projections on two intersecting planes are given.

Planes of Projection.—The fixed planes to which points and lines are referred are known as *planes of projection* or co-ordinate planes.

The two planes are taken at right angles to one another; one is supposed to be horizontal, and is called the horizontal plane or H.P., the other is supposed to be the vertical, and is called the vertical plane or V.P.

Projections.—The projection on the H.P. is called the *plan*, and the projection on the V.P. the *elevation* of the point.

The intersection of the H.P. and V.P. is called the *ground line* (and is denoted by xy); the distance of the plan from xy is the distance of the point from the V.P.; the distance of the elevation from xy is the distance of the point from the H.P.

Angle between a Line and a Plane is the angle between the line and its projection on the plane.

Traces of a Line.—The intersection of a line with the H.P. is called the horizontal trace of the line or H.T.; the intersection with the vertical plane is called the V.T.

Three intersecting co-ordinate planes of projection may be used instead of two planes.

CHAPTER XV.

MENSURATION. PLANE FIGURES AND RIGHT-ANGLED TRIANGLES. AREAS OF RECTANGLE, PARALLELOGRAM, TRIANGLE, RHOMBUS, HEXAGON, QUADRILATERAL, TRAPEZOID, CIRCLE. CIRCUMFERENCE AND AREA OF CIRCLE. SECTOR OF CIRCLE. AREA OF ANNULUS. ELLIPSE.

A parallelogram is a four-sided figure, the opposite sides of which are parallel.

FIG. 105.—A Parallelogram

A rectangle is a parallelogram having each of its angles a right angle

FIG. 106.—A Rectangle.

A square is a parallelogram which has all its sides equal and all its angles right angles.

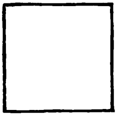


FIG. 107.—A Square.

A rhombus is a parallelogram having all its sides equal, but its angles are not right angles.

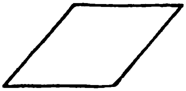


FIG. 108.—A Rhombus.

A trapezium is a four-sided figure which has two of its sides parallel.

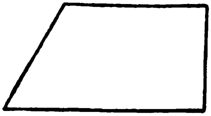


FIG. 109.—A Trapezium.

The **altitude** of a parallelogram is the perpendicular distance between one of the sides assumed as base and the opposite side.

Areas of Plane Figures.—When the number of units of length in two lines at right angles to each other are multiplied together, the product obtained is said to be a quantity of two dimensions, and is referred to as so many square inches, square feet, square centimetres, etc., depending upon the units in which the measures of the lengths are taken. The result of the multiplication gives what is called an **area.**

If the number of units of length in three lines at right angles to one another are multiplied together, the quantity obtained is one of three dimensions and is expressed in cubic inches, cubic feet, or cubic centimetres. The result in such a case is called a **volume.**

It is obvious that although square inches or units of area are derived from, and calculated by means of, linear measure, those quantities only which are of the same dimensions can be added, subtracted, or equated to each other. Thus, we cannot add or subtract a line and an area ; or an area and volume. Results obtained in such cases would be meaningless.

It must be observed, too, that the two lengths multiplied together to obtain the area are *perpendicular to each other.* This applies to the calculation of all areas. If a volume is required the third length must be the line perpendicular to the plane containing the other two.

Unit of Area.—The unit of area is represented by the area of a square, the side of which is of unit length.

To graphically add or subtract the areas of two squares, the sides of which are *AB* and *CD*, *i.e.* to find the side of a square, the area of which will be equal to the sum or difference of the areas of the two given squares, we proceed in the following way.

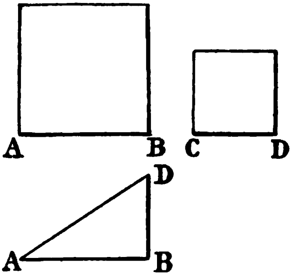


FIG. 110.—Addition and subtraction of areas.

1. **To find their Sum.**—Draw two lines at right angles to one another, as *BD* and *BA* (Fig. 110), making *BD* equal in length to *CD*, and also *BA* and the side of the second square.

Join AD.

Then the square constructed on AD will have an area equal to the sum of the areas of the squares constructed on AB and CD (p. 139).

This is an application of the proposition in Euclid (Book I., Prop. 47) which states that *The square on the hypothenuse of a right-angled triangle is equal to the sum of the squares on the other two sides.*

Or
$$AD^2 = AB^2 + BD^2.$$
$$\therefore \; AD = \sqrt{AB^2 + BD^2}.$$

Ex. 1. Let $AB = 4$ and $BD = 3$.

Then
$$AD = \sqrt{16+9} = \sqrt{25} = 5.$$

2. **To find their Difference.**—Let AB and CD (Fig. 110) be the sides of two given squares. To find the side of a square, the area of which will be equal to the difference of the areas of the two squares.

Draw two lines at right angles to one another, as AB and BD making BD equal in length to CD, the side of the smaller square.

With D as centre, and a radius AD equal to AB, the side of the larger square, describe an arc of a circle.

Let A be the point of intersection of the arc with the horizontal line BA.

Then BA is the length of the side of the square required.

Ex. 2. Let $AB = 10$ cm. and $DC = 8$ cm.

As before draw two lines at right angles making BD equal to 8 cm. With D as centre, radius 10 cm., describe an arc of a circle cutting AB at A.

Then AB is the length required.
$$AB = \sqrt{10^2 - 8^2} = \sqrt{36} = 6.$$

Area of a Rectangle.—*The number of units of area in a rectangular figure is found by multiplying together the numbers of units of length in two adjacent sides.*

Thus, if AB and BC (Fig. 111) are two adjacent sides of a rectangle; the area is the product of the number of units of length in AB, by the number of units of length in BC.

The number of units in the lowest line of the figure AB is usually called the *base*, or length, and the units in line BC, perpendicular to this, the *altitude*, or breadth.

Hence $\quad$ area = *base* × *altitude*,

or $\quad$ = length × breadth = ab.

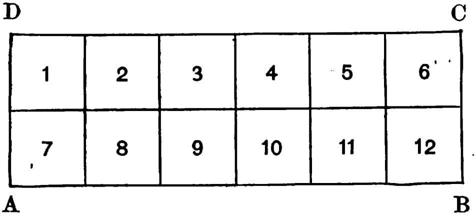


FIG. 111.—Area of a rectangle.

Ex. 1. If the base AB and height BC are 6 and 2 units length respectively, the area is 12 square units. If AB be divided into 6 equal parts and BC into 2, then by drawing lines through the points of division parallel to AB and BC, the rectangle is seen to be divided into 12 equal squares (Fig. 111).

The area is obtained in a similar manner when the two given numbers denoting the lengths of the sides are not whole numbers.

Ex. Obtain the area of a rectangle when the two adjacent sides are 5 ft. 9 in. and 2 ft. 6 in. in length respectively.

We may reduce to inches before multiplying.

Thus $\quad$ 5 ft. 9 in. = 69 inches

and $\quad$ 2 ft. 6 in. = 30 inches;

∴ area of rectangle = 69 × 30 square inches

= 2070 square inches = 14·375 square feet;

or, instead of first reducing feet to inches and afterwards multiplying, we may proceed as follows:

5 ft. 9 in. $= 5\frac{3}{4}$ feet and 2 ft. 6 in. $= 2\frac{1}{2}$ feet;

∴ area of rectangle $= 5\frac{3}{4} \times 2\frac{1}{2}$ square feet

$= \frac{23}{4} \times \frac{5}{2}$ square feet $= 14\frac{3}{8}$ square feet.

It is an easy matter to verify the latter example as shown in Fig. 112, where $AB = 5\frac{3}{4}$ ft. and $BC = 2\frac{1}{2}$ ft.

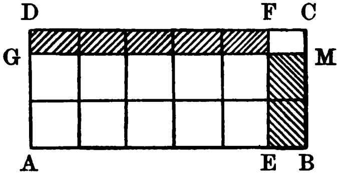


FIG. 112.—Area of a rectangle.

The rectangle $AEG = 10$ square feet.

Area of the two rectangles $EBM = 2(1 \times \frac{3}{4}) = 1\frac{1}{2}$ square feet.

" the five rectangles $DFG = 5(1 \times \frac{1}{2}) = 2\frac{1}{2}$ "

" rectangle $FCM = \frac{3}{4} \times \frac{1}{2} = \frac{3}{8}$.

Hence area of rectangle $ABCD = 10 + 1\frac{1}{2} + 2\frac{1}{2} + \frac{3}{8}$ square feet

$= 14\frac{3}{8}$ square feet.

Area of a Parallelogram.—As already indicated on p. 229, the rectangle is a particular case of the parallelogram, hence

area of parallelogram = *base* × *altitude*.

This may also be shown as follows:

Let $ABCD$ (Fig. 113) be the given parallelogram.

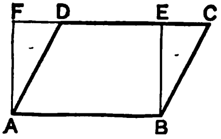


FIG. 113.—Area of a parallelogram.

Draw AF and BE perpendicular to AB, and meeting CD at E, and CD produced at F, then (by Euclid, Bk. I., Prop. 35), the rectangle $ABEF$ is equal to the parallelogram $ABCD$.

Hence, area of parallelogram

$= base \times altitude = ab.$

As $b = AD \sin EDA$,

the area $= AB \times AD \sin EDA$,

or the *product of two adjacent sides and the sine of the included angle gives the area of a parallelogram.*

Ex. 1. If the base $AB = 4$ ft. and the altitude $BE = 1\frac{1}{2}$ ft.

$\therefore$ area $= 4 \times 1\frac{1}{2} = 6$ sq. ft.

Ex. 2. The two adjacent sides of a parallelogram are 10 ft. and 8 ft. respectively, and the included angle 30°; find the area.

Area $= 10 \times 8 \times \sin 30° = 10 \times 8 \times \frac{1}{2} = 40$ sq. ft.

If the two figures shown be drawn on a sheet of paper, then by cutting out the right-angled triangle *BCE*, inverting and fitting it to the other end as shown, the parallelogram is converted into a rectangle.

By means of a simple piece of apparatus formed of a number of rectangular pieces of millboard, cardboard, or thin wood, a rectangle can be formed the area of which is equal to $AB \times BE$ units of area, shown by the rectangle *ABEF* (Fig. 113).

These may be made to slide over each other, forming a parallelogram *ABCD* (Fig. 113), and the area is obviously unaltered by merely changing the position of the pieces, the base and the height remaining unaltered.

EXERCISES. LX.

1. The sides of a parallelogram, including an angle 60°, are 20 and 30; and the sides of another, including an angle 120°, are 30 and 40. Compare their areas.

2. The sides of a four-sided figure taken in order are 135, 18, 150, and 125 feet, and the angle contained by the first two is a right angle. Determine the area of the figure.

3. The diagonals of a parallelogram are 6 and 8 ft. long, and include an angle of 67°. Find the length of the shorter sides of the parallelogram.

4. A rectangular area of 8 ac. 480 yds. has a perimeter of 840 yds. Find its length and breadth.

5. What is the cost of a door measuring 6 ft. 3 in. by 3 ft. 6 in. at 1s. 8d. per square foot?

6. The area of a rectangular field is 14 a. 2 r. 11 p. What is its length, its breadth being 925 links?

7. The sides of a parallelogram are as 3 : 2, and the angle between them is half a right angle. If the area be $\frac{11}{6}$ sq. ft. find the two sides correct to two decimal places.

8. Find the area of a square slab, the side of which measures ·2868 of a metre.

9. Find the number of square feet of glass in a window which measures 7 feet 10¾ inches by 4 feet 7½ inches.

10. Find the area of a rectangle 2 ft. 3 in. by 5 ft. 7 in., and find the cost of glazing a window containing 60 panes, each of which is 2′ 3″ by 11′ 5″, at 3s. 8d. per square foot.

11. A plot of land measures 300 acres, and is in the form of a rectangle having its length three times its breadth. Find to the nearest yard the length of fence which would be required to enclose it.

12. A blackboard 4 ft. 6 in. wide rests with its lower edge parallel to and 1 ft. 6 in. from a wall, its upper edge touches the wall 4 ft. from the ground. Find the area of the blackboard.

13. Calculate how many acres there are in a square mile.

14. A postage stamp is $\frac{23}{24}$ in. long and $\frac{4}{5}$ in. wide; how many will be required to cover a board 1 ft. 11 in. long and 1 ft. wide?

Area of a Triangle.—In Fig. 114 the rectangle $ABCD$ and the parallelogram $ABEF$ on the same base AB and of the same altitude, are equal in area.

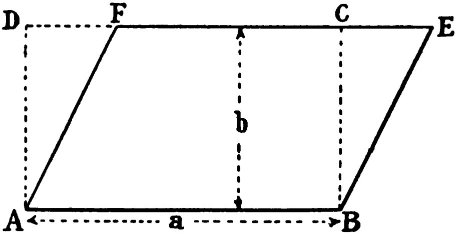


FIG. 114.—Area of a triangle.

If A were joined to C and to E.

Then it is easily seen that the triangle ACB is half the rectangle $ABCD$, and the triangle AEB is half the parallelogram $ABEF$ (p. 36).

Hence the two triangles are equal in area, and the area in each case is equal to *half the product of the base and the altitude.*

$$\therefore \text{ area of triangle} = \tfrac{1}{2}(\text{base} \times \text{altitude}) = \tfrac{1}{2}ab.$$

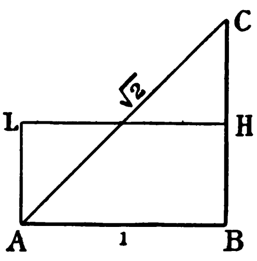


FIG. 115.—Area of an isosceles triangle.

This result may be shown graphically by the rectangle $ABHL$ (Fig. 115) on the same, or an equal base, and of half the height.

If in any triangle ABC (Fig. 115) the height a and an angle BAC be given. Then as $a = AC \sin BAC$ (p. 159) the area can be expressed in the following convenient form:

$$\text{area} = \tfrac{1}{2}(AB \times a)$$
$$= \tfrac{1}{2}(AB \times AC \sin BAC).$$

The angles of a triangle are usually denoted by the letters A, B, C, and the sides of the triangle opposite the angles by a, b, c.

Hence, area of triangle $=\frac{1}{2}bc\sin A$.

When the triangle is right-angled the results are very important, and may, if necessary, be committed to memory.

If a right-angled triangle is also isosceles, so that $AB=BC$ (Fig. 115) then denoting AB and BC each by unity,

$$AC=\sqrt{1^2+1^2}=\sqrt{2}.$$

If the equal sides are unit length, the area is

$$\frac{1\times 1}{2}=\frac{1}{2} \text{ square unit.}$$

In an *equilateral triangle* the three sides are all of equal length, and each angle is 60°.

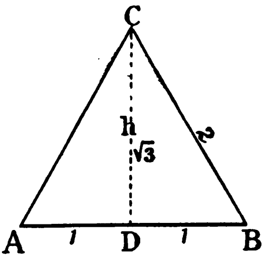


FIG. 116.—Equilateral triangle.

If, as shown (Fig. 116), a line CD be drawn perpendicular to the base, and meeting the base AB at D, then point D bisects the base so that $AD=DB$ (p. 35).

If each of the equal sides be two units in length,

$$\therefore\ AD=1,$$

and $$CD=\sqrt{2^2-1^2}=\sqrt{3}.$$

So that in a right-angled triangle in which one angle is 60°, and the other 30°, the three sides of the triangle are therefore proportional to 2, 1, and $\sqrt{3}$.

Also, area of triangle $ABC=\frac{1}{2}\times 2\times\sqrt{3}=\sqrt{3}$ square units.

When the three sides of a triangle are given:

Let a, b, c be the three sides and $s=\frac{a+b+c}{2}$; when $s=$half the sum of the sides, then the area is given by the formula

$$\text{area}=\sqrt{s(s-a)(s-b)(s-c)},$$

or, find half the sum of the sides, subtract from this half sum each side separately, multiply the three remainders and the half sum together; the square root of the product is the area required.

To obtain this expression for the area of a triangle we may proceed as follows:

Let ABC (Fig. 117) be a given triangle; from the vertex C let fall a perpendicular CD on the base AB.

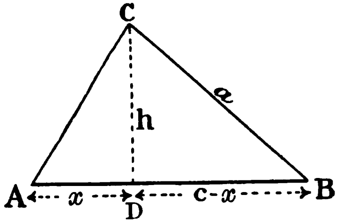


FIG. 117.

Denoting in the usual way the side AB by c, then if $AD=x$, $DB=c-x$.

From the right-angled triangle ADC we have

$$CD^2=b^2-x^2.$$

Also from the right-angled triangle BDC.

$$CD^2=h^2=a^2-(c-x)^2;$$

$$\therefore\ b^2-x^2=a^2-(c-x)^2;$$

$$\therefore\ x=\frac{b^2+c^2-a^2}{2c}. \quad \text{.........(1)}$$

Now, area of triangle $ABC=\frac{1}{2}bc\sin A$,

where
$$\sin A=\frac{CD}{b}=\frac{\sqrt{(b^2-x^2)}}{b}$$
$$=\frac{\sqrt{(b-x)(b+x)}}{b};$$

$$\therefore\ \text{area of triangle}=\frac{\frac{1}{2}bc\sqrt{(b-x)(b+x)}}{b}. \quad \text{....................(2)}$$

Substituting in (2) the value of x from (1),

$$\text{area of triangle}=\frac{c}{2}\sqrt{\frac{(2bc-b^2-c^2+a^2)}{2c}\,\frac{(2bc+b^2+c^2-a^2)}{2c}}$$
$$=\tfrac{1}{4}\sqrt{\{a^2-(b-c)^2\times(b+c)^2-a^2\}}$$
$$=\tfrac{1}{4}\sqrt{(a+b-c)(a-b+c)(b+c+a)(b+c-a)}. \quad \text{...(3)}$$

Let s denote $\frac{a+b+c}{2}$.

Then
$$a+b+c=2s$$
$$a+b-c=2s-2c$$
$$a+c-b=2s-2b$$
$$b+c-a=2s-2a.$$

Substituting these values in (3)

$$\text{area of triangle}=\tfrac{1}{4}\sqrt{2s(2s-2a)(2s-2b)(2s-2c)}$$
$$=\tfrac{1}{4}\sqrt{16s(s-a)(s-b)(s-c)}$$
$$=\sqrt{s(s-a)(s-b)(s-c)}.$$

Ex. 1. In a triangle the lengths of the three sides are 6, 8, and 10 ft. respectively. Find the area.

$$s=\tfrac{1}{2}(6+8+10)=12$$

$$\therefore\ \text{area}=\sqrt{12(12-6)(12-8)(12-10)}$$

$$=\sqrt{12\times6\times4\times2}=\sqrt{576}=24 \text{ sq. ft.}$$

EXERCISES. LXI.

1. One side of a triangle is 200 yds., another is 88 yds.; determine the third side, when the area is 5095 sq. yds.

2. Find the area of a triangle whose sides are 16·1, 23·45, 29·53 respectively.

3. Find the area of a triangular field whose sides are 7·32, 4·57, and 5·48 chains respectively.

4. The sides of a triangular field are 10·42, 8·74, and 12·63 chains respectively; find the area.

5. Find the area of a triangle, the sides of which are 105 feet, 116 feet, and 143 feet.

6. If a triangle is equilateral, length of side a, show that its area

$$=\frac{a^2\sqrt{3}}{4}.$$

7. If a triangle is isosceles, side $a=b$, show that the area is

$$\frac{c}{4}\sqrt{4a^2-c^2}.$$

8. Find area of a triangle, sides 5, 9, and 8 ft. respectively.

9. Find the area of the triangle whose sides are respectively 384·11, 564·67, and 663·44 units.

10. A triangle has its three sides 800, 780, and 500 feet respectively; find the side of a square equal to it in area.

11. Find the area of an equilateral triangle, length of side 38 ft.

12. What is the area of a grass plot in the form of an equilateral triangle, each side 36 ft.?

13. The sides of a triangular field are 181, 259, and 300 ft. respectively. If its rent be 10s. 9½d., what is the rent per acre?

14. Find the area of a triangle whose sides are 8 yds., 7 ft., and 25 ft.

Area of a Rhombus.—Let $ABCD$ (Fig. 118) be a given rhombus; by joining A to C, the figure is divided into two triangles ACB and ACD, also as *the two diagonals of a rhombus bisect each other at right angles* (p. 35), the line IB is the altitude of the triangle ACB and ID is the altitude of the triangle ACD.

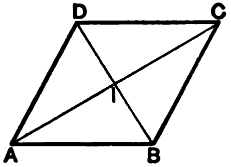


FIG. 118.—Area of a rhombus.

Area of rhombus

$$= \text{area of triangles } ACD \text{ and } ACB.$$
$$= \tfrac{1}{2}(AC \times IB) + \tfrac{1}{2}(AC \times ID),$$
$$= \tfrac{1}{2}(AC \times BD),$$

hence *the area of a rhombus is half the product of the two diagonals.*

Ex. 1. Find area of a rhombus whose diagonals are 25 and 15 feet respectively.

$$\text{Area} = \tfrac{1}{2}(25 \times 15) = 187 \cdot 5 \text{ square feet.}$$

Area of any Quadrilateral.—Any four-sided figure, such as $ABCD$ (Fig. 119) is called a quadrilateral. By joining D to B the figure is divided into two triangles ADB and DBC. The area of the quadrilateral is the sum of the areas of the two triangles. Draw AE and CF perpendicular to DB.

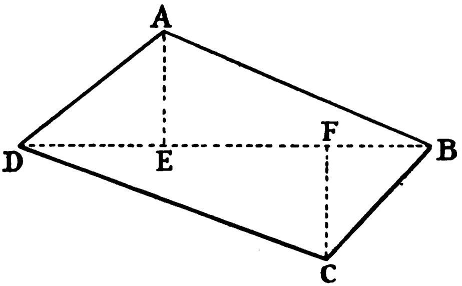


FIG. 119.—Area of a quadrilateral.

Then area of triangle $ADB = \tfrac{1}{2}(DB \times EA)$,
Also area of triangle $DCB = \tfrac{1}{2}(DB \times CF)$,
$\therefore$ area of quadrilateral $= \tfrac{1}{2}DB(EA + CF)$,

or, *half the product of one of the diagonals and the sum of the two perpendiculars let fall on that diagonal from the two opposite angles gives the area of a quadrilateral.*

Area of a Polygon.—In a similar manner the area of any irregular polygon may be found by dividing the figure into a number of triangles, finding the area of each triangle, and finally adding together all the areas so obtained. The result is the area of the figure.

Graphically.—The area of an irregular figure may be obtained graphically as follows:

Let *ABCDE* (Fig. 120) be an irregular five-sided figure.

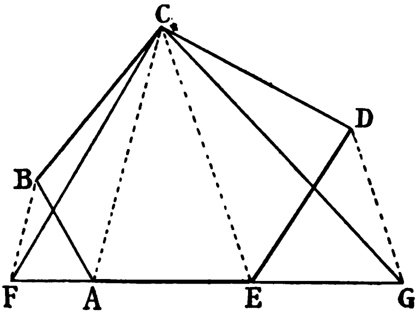


FIG. 120.

Join *C* to *A* and from *B* draw a line *BF* parallel to *CA*, meeting *EA* produced at *F*. Join *C* to *F*.

The triangles *CFA* and *CBA* are on the same base *CA* and have the same altitude. Hence the triangle *CFA* is equal in area to the triangle *CBA*.

In a similar manner, join *C* to *E*, and from point *D* draw *DG* parallel to *CE*, meeting *AE* produced at *G*. Join *C* to *G*.

The triangles *CGE* and *CDE* are equal in area.

Thus the triangle *FCG* is equal in area to the given five-sided figure.

The triangle *FCG* may be converted into an equivalent rectangle as on p. 234.

Area of a Trapezium.—The area may be obtained by dividing the figure into two triangles ACD and ACB (Fig. 121).

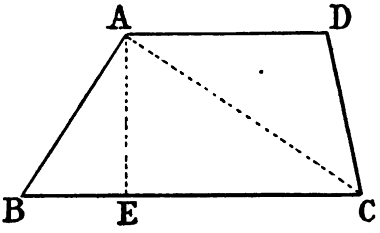


FIG. 121.—Area of a trapezium.

If AE be drawn perpendicular to BC, and meeting BC in E, then AE is the altitude of both the triangles ABC and ACD.

$\therefore$ area of triangle $ABC = \frac{1}{2}(BC \times AE)$,

area of triangle $ACD = \frac{1}{2}(AD \times AE)$.

Area of four-sided figure $ABCD$ equals area of the two triangles into which the figure is divided by the line AC.

$$\therefore \text{ area of } ABCD = \tfrac{1}{2}(BC + AE) + \tfrac{1}{2}(AD \times AE),$$
$$= \tfrac{1}{2}AE\,(BC + AD).$$

This important result for the *area of a trapezium* may be stated as follows: *Multiply the sum of the parallel sides by half the perpendicular distance between them.*

.EXERCISES. LXII.

1. The parallel sides of a trapezium are 8 ft. and 3 ft.; its two other sides are 7 ft. and 5 ft.; find its area.

2. The diagonals of a rhombus are 8·36 and 7·2 in. respectively. Find its area.

3. The parallel sides of a trapezium are 234 and 104 ft. respectively, and the perpendicular distance between them is 92 ft., find the area.

4. The parallel sides of a trapezium are 54 ft. and 36 ft., and perpendicular distance between them is 10 ft. Find the area.

5. Find the acreage of a trapezoidal field, the width being 2150 links and the parallel sides 1856 and 1623 links.

6. Each side of a rhombus is 60 ft., and one of the diagonals 100 ft. Find the area.

7. Find the side of a hexagon, the area being 300 sq. yds.

8. In a map of a country a district is found to have two of its boundaries very approximately parallel and equal to 276 miles and 216 miles; if the breadth be 100 miles find the area.

9. Each side of a rhombus is 60 ft., and one of the diagonals 100 ft. Find the area.

10. Find the area of a field in the form of a trapezoid, parallel sides 745 and 749 links, and perpendicular distance between them 1379 links.

Circumference of a Circle.—The number of times that the circumference of a circle contains the length of the diameter of a circle cannot be expressed exactly, but it is very nearly 3·14159265. The number 3·1416 is used for convenience and is sufficiently exact for nearly all purposes. This number is denoted by the Greek letter π.

An approximate value of π, sufficiently exact for all practical purposes, and very convenient when four-figure logarithms are used, is $3\frac{1}{7}$ or 3·142. Thus, $\pi = 3{\cdot}14159265 = \frac{22}{7}$ within $\frac{1}{20}$ per cent.

That the circumference of a circle is πd where d is the diameter may be shown in several ways; two simple experimental methods, as follows, will be sufficient in this place.

1. By rolling a disc of metal or wood on any convenient scale. Make a mark on the circumference of the disc, put the mark coincident with a scale division, roll the disc along the scale until the mark is again coincident with the scale, and note carefully the distance in scale divisions moved through. Then, by applying the scale to the disc obtain its diameter.

Simple division will show that the circumference is $3\frac{1}{7}$ times the diameter.

2. Or, wrap a piece of thin paper round the disc and mark, by two points, the line along which the edges overlap; unroll the paper, and its length when measured will be found to be $3\frac{1}{7}$ times the diameter.

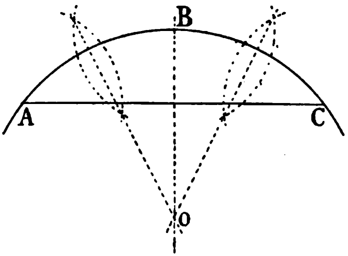


Fig. 122.—Circular arc.

Length of a Circular Arc.—Let ABC (Fig. 122) be a circular arc, it is required to find the centre of the circle of which ABC is a portion.

With A and B as centres and any convenient radius, describe arcs intersecting as shown; a line drawn through the intersections of these arcs will, if produced, pass through the centre of the circle.

Similarly, by using B and C as centres another line is obtained which also passes through the centre. The point O or intersection of the two lines is the centre of the circle required.

Let the radius be denoted by r.

Join O to A and C. Call the angle AOC, θ.

Then the measure in **radians** (p. 30) of the angle $\theta = \frac{\text{arc } AC}{r}$;

$$\therefore \text{ arc } AC = r \cdot \theta.$$

Thus, when the circular measure of the angle and the radius of the circle are known the arc can be calculated.

If the angle is given in degrees it is necessary to convert to circular measure; if N denote the number of degrees in an angle, then $\theta = N \times \frac{\pi}{180}$, that is N divided by the unit angle 57·3.

Rankine's Approximations.—By methods due to the late Prof. Rankine it is possible *to set off a line AT* (Fig. 123) *equal in length to a circular arc AB.*

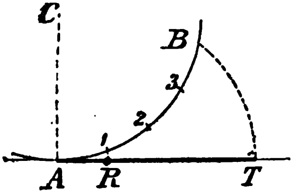


FIG. 123.

Draw a tangent to the arc at A. Divide the arc AB into four equal parts as shown.

With centre A radius $A1$ describe an arc cutting the tangent at R. With centre R, radius RB, describe arc. Then AT = arc AB approximately.

To set off an arc AB equal to a given straight line AT. Divide AT into four equal parts, and using the first point R as centre and radius RT describe arc TB. Then arc AB = straight line AT. To obtain a good approximation the angle TAC should not exceed 90°. The error consists in the line being a little shorter than the corresponding arc; the amount about 1 in 900 for an arc which subtends an angle of 60°.

Ellipse.—It is only possible to obtain a rough approximation to the circumference of an ellipse. Perhaps the best rule is;

$$\text{circumference of ellipse} = \frac{(d_1 + d_2)}{2} \times \pi,$$

where d_1, d_2 are the major and minor axes respectively.

Or, in words, to obtain the circumference of an ellipse, *multiply half the sum of the major and minor axes by* π.

Ex. 1. Find the circumference of an ellipse in which the major axis is 8 inches and the minor axis 6 inches.

$$\text{Half the sum} = \frac{8+6}{2} = 7.$$

$$\therefore \text{ circumference} = 7 \times \pi = 22 \text{ inches.}$$

EXERCISES. LXIII.

1. A well is 100 feet deep; how many coils of rope will be required to reach to the bottom, the roller on which they are wrapped being 8 inches in diameter?

2. If the metre is equivalent to 39·37009 inches English, and corresponds to the ten millionth part of the distance from the pole of the earth to the equator; find the circumference of the earth in miles, etc.

3. The diameter of a valve is 3½ inches, find its circumference.

4. Find length of the outer ring of a paddle-wheel, diameter 26 feet.

5. The circumference of a circle being 50 feet, find the diameter.

6. Find the diameter of the pitch circle of a wheel of 120 teeth, the pitch (distance between two consecutive teeth) being 2 inches.

7. The circumference of a piston is 226·19 inches; find the diameter.

8. A wheel 30 inches diameter is found to have made, during a journey, 67,802 revolutions. Find the length of the journey assuming no slipping to occur.

9. The radius of a circle is 10 feet; express in degrees the angle subtended at the centre by an arc 3 feet in length.

10. The radius of a circle is 105 feet; find the length of arc which subtends an angle of 37° 15′ at the centre.

11. How much land will be required for a road 30 feet wide around an elliptical plot of ground, the axes of the ellipse being 200 and 170 yards respectively?

Area of a Circle.—If a regular polygon be inscribed in a circle, a series of triangles are formed by joining the points to the centre.

The area of each little triangle is one-half the product of the base and the perpendicular let fall from the centre on the base.

The length of the base may be denoted by a and the perpendicular by p; also let r denote the radius of the circle. The area of the polygon will be $\frac{1}{2}p(a+a+\dots)$ or $\frac{1}{2}p\Sigma a$.

As the number of sides in the polygon is increased, its area becomes nearer and nearer that of the circle, and when the number of sides is indefinitely increased, the perimeter (or sum of sides) of polygon becomes equal to the circumference of the circle $=2\pi r$;—the perpendicular also becomes the radius of the circle. Hence the area $=\frac{1}{2}(2\pi r\times r)=\pi r^2$.

By dividing the circle into a large number of sectors, the bases may be made to differ as little as possible from straight lines. Each of the sectors forming the lower half of the circumference could be placed, along a horizontal line AB (Fig. 124). A corresponding number of sectors from the upper half of the circumference could be placed along the upper line CD, completing the rectangle $ABCD$. The base AB will then be half the circumference of the circle and the height of the rectangle is equal to r.

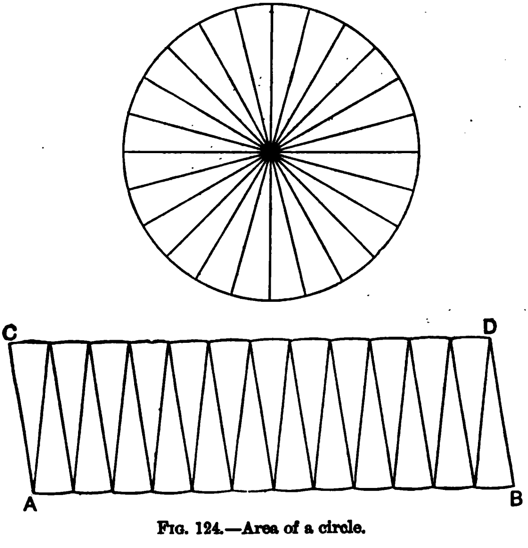


FIG. 124.—Area of a circle.

∴ Area of rectangle $= AB \times r = r \times \frac{1}{2} 2\pi r = \pi r^2$.

Experimental Method. — A good experimental method where a balance is available is to draw two circles on a sheet of cardboard and to circumscribe one of the circles by a square $ABCD$ as shown in Fig. 125. By drawing two lines at right angles and passing through the centre of the circle, the square is divided into four equal squares, the length of side of each being r.

Make $EM = \frac{1}{7}$ of ED.

Draw MN parallel to AD.

Then the area of the shaded figure $= 3\frac{1}{7} r^2$. Cut out a circle and also the shaded portion. If this be done with care the weights of the two will be found to be the same.

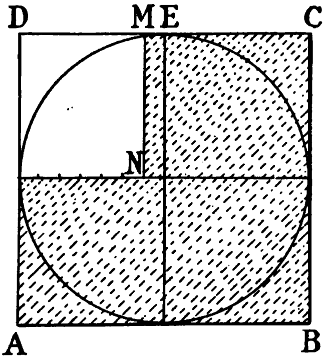


FIG. 125.—Area of a circle experimentally.

Using Rankine's Approximation.—As the area of a circle can be expressed in the form $\pi r \times r$, i.e. *half the circumference of a circle multiplied by the radius*, we can use the approximation due to Rankine to obtain the area of a circle graphically.

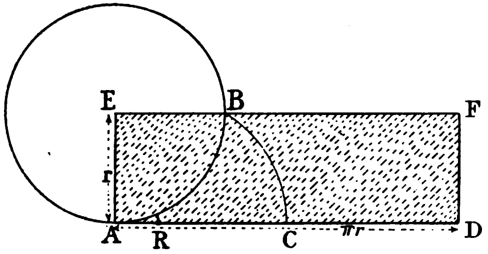


FIG. 126.—Area of a circle obtained graphically.

Thus, as shown in Fig. 126, make $AC =$ quadrant AB.

Produce AC to D, making $CD = AC$.

Then $AD = \pi r$.

Complete the rectangle *ADFE* having its adjacent sides *AD* and *DF* equal to πr and r respectively.

Then area of circle is equal to area of rectangle

$$ADFE = \pi r \times r = \pi r^2.$$

Ex. 1. If the diameter of a piston is 30 inches, that of the rod being 3 inches (Fig. 127), find the total pressure on the piston.

Pressure of steam

$= 100$ lbs. per sq. in.;

area $= \frac{\pi}{4} \times 30^2 = 706{\cdot}86$ sq. in.;

$\therefore$ total pressure on piston

$= 100 \times 706{\cdot}86 = 70686$ lbs.

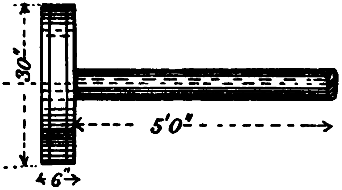


FIG. 127.

Ex. 2. The diameter of the safety-valve in a boiler is 3 inches; find the total pressure tending to raise the valve when the pressure of the steam is 120 lbs. per sq. in.

$$\text{Area of valve} = \frac{\pi}{4} \times 3^2 = 7{\cdot}07 \text{ sq. in.};$$

$$\therefore \text{ total pressure} = 7{\cdot}07 \times 120 = 848{\cdot}4 \text{ lbs.}$$

EXERCISES. LXIV.

1. Find the area of an air-pump bucket which is $\frac{1}{5}$ that of the cylinder, the diameter of the cylinder being 63 inches.

2. What is the diameter of a valve containing 100 square inches?

3. A circular field 1000 feet in diameter is surrounded by a plantation 10 feet wide. Find (i) areas of field and plantation; (ii) circumferences of field and plantation.

4. A circular bowling-green, of which the diameter is 125 feet, is surrounded on the outside by a gravel path $7\frac{1}{2}$ feet wide; find the area of the path in square yards.

5. If a pressure of 15 lbs. to the square inch be applied to a circular plate 3 ft. in diameter, what is the total pressure?

6. If the pressure of steam in a boiler is 100 lbs. per sq. in., find the total pressure on a valve 3 inches diameter.

7. If the diameter of a piston is 12 inches, find the total pressure on the piston when the steam pressure is 100 lbs. per sq. in.

8. Find the areas of circles whose diameters in inches are as follows: $2{\cdot}5$; $\cdot 0555$; $\cdot 3725$; $\frac{1}{32}$; $\frac{3}{8}$; $\frac{1}{4}$; $\cdot 01$; $\cdot 07$.

Area of Sector of a Circle.—*To find the area of the sector AE* (Fig. 128) *the angle θ being known.*

As the whole circle consists of 360 degrees, or 2π radians, the area of sector will be the same fractional part of the whole area that the angle θ is of 360°, or of 2π.

Denoting the angle in degrees by N, then

$$\text{area of sector} = \frac{N}{360}\pi r^2 = \frac{\theta}{2\pi}\pi r^2 = \frac{\theta r^2}{2}.$$

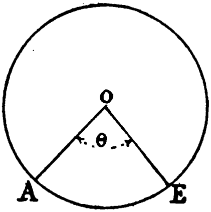


FIG. 128.—Area of sector of a circle.

Ex. 1. Find the area of the sector of a circle containing an angle of 42°, the radius of the circle being 15 feet;

$$\text{area of circle} = \pi \times 15^2;$$

$$\text{area of sector} = \frac{42}{360}\pi \times 15^2 = 82{\cdot}46 \text{ square feet.}$$

EXERCISES. LXV.

1. Find the area of a sector of a circle the angle of which is 60°, the radius of the circle being 60 ft.

2. A sector of a circle contains an angle of 45°, and the area of the sector is 200 square yds.; find the radius.

3. A field is in the form of a sector of a circle, its angle is 125° 15′, and radius 100 ft.; find the length of the arc of the circle, and its area.

4. Find the area of a circular segment whose height is 7 in. and base 3 ft.

5. Find the area of the sector of a circle whose arc of 20° is 18 in. long.

6. The radius of a circle is 20 in.; find the area of the remaining portion when a segment having an arc of 25° has been cut off from it.

7. The arc of a circle measures 10 ft., and subtends an angle of 24°; find the area of the sector.

8. The radius of a circle being known, give the formulae for the finding of its circumference and its area. If the radius of a circle be 25 ft., find the number of square feet in a sector the arc of which subtends an angle of 157° 30′.

Area of an Annulus.—If R (Fig. 129) denote the radius of the outer circle, and r the radius of the inner; the area of the annulus is the difference of the two areas;

$$=\pi R^2-\pi r^2$$

$$=\pi(R^2-r^2)=\pi(R+r)(R-r);$$

∴ *multiply the sum and difference of the two radii by* $3\frac{1}{7}$ *to obtain the area of an annulus.*

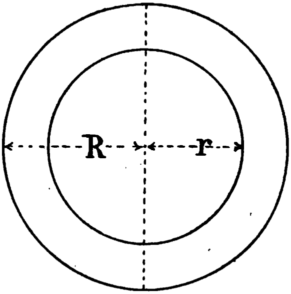


FIG. 129.—Annulus.

Ex. 1. Calculate the effective pressure on a piston 36 inches diameter (Fig. 130); the diameter of piston rod = 6 inches; and the pressure of steam = 160 lbs. per sq. in.

area of one side the piston

$$=\frac{\pi}{4}(36)^2=1017{\cdot}9,$$

total pressure

$$=1017{\cdot}9\times 160;$$

area of the other side

$$=\frac{\pi}{4}(36^2-6^2)=989{\cdot}63;$$

$$\therefore\ \text{total pressure}=989{\cdot}63\times 160,$$

or the difference in the total pressures on the two sides of the piston is $160(1017{\cdot}9-989{\cdot}63)=4523{\cdot}8$ lbs.

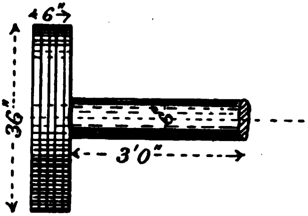


FIG. 130.

Area of an Ellipse.—The two axes AA and BB' of an ellipse (Fig. 131) are called the major and minor axis respectively. Denoting these by $2a$ and $2b$, it is easy to show by using an experimental method similar to that adopted in finding the area of a circle that the *area of an ellipse* $=\pi\times a\times b$.

Draw on a sheet of cardboard two ellipses axes $2a$, and $2b$, circumscribe one of them by a rectangle, and draw the two axes AA and BB'. Make $BM=\frac{1}{7}BD$, and draw MN parallel to ED.

Each of the three rectangles is $a\times b$ and as $BMNO$ is $\frac{1}{7}ab$, the area of the shaded portion is $3\frac{1}{7}ab$.

If this area be cut out carefully and placed in one pan of a balance, while an ellipse of the same size, and cut out of the same material, be placed in the other pan, the two will be found to balance.

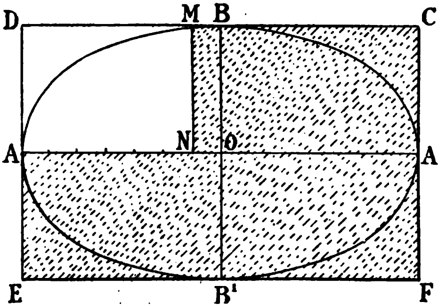


FIG. 131.—Area of an ellipse obtained experimentally.

Another method of obtaining the area is to draw the ellipse on squared paper. Count the enclosed squares, also estimate as accurately as possible the total area of the squares partially cut by the profile of the figure. By adding the two together, the area can be found.

EXERCISES. LXVI.

1. The greater and lesser diameters of an elliptical manhole door are 2 feet 9 inches and 2 feet 6 inches; find its area.

2. What is the area of an elliptical door whose dimensions are 12 inches by 9 inches?

3. The greater diameter of an elliptical funnel is 4 feet 6 inches, and the lesser diameter 4 feet; what space will it occupy on the deck, and how many square feet of iron does it contain if its height be 16 feet?

4. Find the difference in area of two fields, the first in form of a circle whose radius = 575 yards, and the other in form of an ellipse whose axes are 1000 yards and 1250 yards.

5. The semi-axes of an ellipse are 12 and 7 inches respectively. A circle is described on the minor axis. What is the area of that part of the ellipse without the circle.

6. The diameter of a piston is 25 inches, the piston-rod $4\frac{7}{8}$ inches; find the difference in the total pressure on both sides of the piston if the pressure of the steam be 160 lbs. per square inch.

7. The external and internal diameters of a hollow steel shaft are 16 and 8 inches respectively. Find the area of the cross-section.

8. If the pressure of steam is 100 lbs. per square inch, find the difference in the effective pressure on the two sides of the piston (Fig. 127).

Summary.

Area of a Rectangle = *base* × *altitude* = *length* × *breadth.*

Area of a Parallelogram = *base* × *altitude* = *half the product of adjacent sides into the sine of included angle.*

Area of a Triangle = $\frac{1}{2}$ base × altitude; or, half the product of two sides by the sine of included angle; or, from half the sum of the three sides subtract each side separately. Multiply the half sum and the three remainders together and find the square root of the product.

Area of a Rhombus = *half the product of the two diagonals.*

Area of a Trapezium = Multiply *half the sum of the parallel sides by the perpendicular distance between them.*

Circumference of a Circle = $\pi \times$ (diameter) = 2π(radius), where π denotes 3·1416 or $\frac{22}{7}$, approximately.

Area of a Circle = π(radius)^2 = $\frac{\pi}{4}$(diameter)^2.

Area of Sector of a Circle = $\frac{\theta}{2}r^2$, where θ is the angle in radians subtended at the centre.

Area of an Annulus = $\pi(R^2 - r^2)$, or subtract the square of the smaller radius from the square of the greater and multiply the difference by π.

Area of an Ellipse = $\pi \times ab$ = product of its semi-axes multiplied by π.

CHAPTER XVI.

SIMPSON'S RULE.

Areas of an irregular Figure.—To find the area of figure *ABDG*, of which one of the boundaries is curved.

The usual way of estimating approximately the area of a curve is to divide the base *AB* (Fig. 132) into a number of equal parts, and to erect ordinates at the centre of each part, as shown.

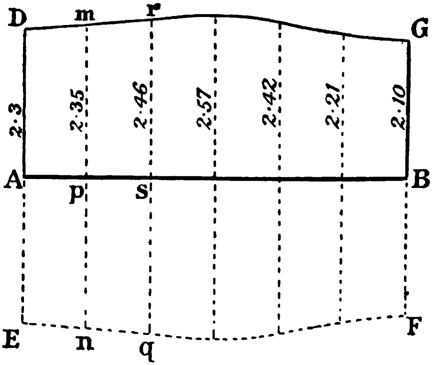


FIG. 132.

The length of each perpendicular from the base *AB* to the point where the vertical cuts the curve is carefully measured, and all the ordinates added together, the sum so obtained is divided by the number of ordinates. In this way the mean ordinate may be obtained.

A very convenient method of adding the ordinates is to mark them on a slip of paper, adding one to the end of the other until the total length is determined. The mean ordinate gives the mean height h approximately.

The approximation depends upon the number of ordinates taken. The approximation more closely approaches the actual value the greater the number of ordinates used.

The product of the mean ordinate and the base is the area required. For comparatively small diagrams, ten strips are usually taken. This number is sufficiently large to give a fair average, and, moreover, dividing by 10 can be effected by merely shifting the decimal point.

Thus, in estimating approximately the mean height of any *diagram* (Fig 133), the base GF is divided into a number of equal parts, usually 10. The height of the ordinate raised at the centre of each part is measured, the mean of these is approximately the mean height of the curve.

The length GF may correspond, on a reduced scale, to the travel of the piston in a cylinder, and the ordinates of the curve represent to a known scale the pressure per sq. inch of the steam in the cylinder at the various points of the stroke.

Hence the mean height (Fig. 201) indicates the mean pressure P of the steam in pounds per sq. inch throughout the stroke (the stroke being the term applied to the distance moved through by the piston in moving from its extreme position at one end of the cylinder to a corresponding position at the other end).

If A denote the area of the piston in square inches, then the total force is $P \times A$, and the work done by this force in acting through a length of stroke L is $P \times A \times L$. If N denote the number of strokes per minute, the work done per minute $= PALN$.

But the unit of power used by engineers is 33000 ft. lbs. per minute, and called a Horse-power.

Hence horse-power of the engine $= \dfrac{P \times L \times A \times N}{33000}$

The Trapezoidal Rule is another method to obtain the mean ordinate h, having divided the base into a number of equal parts, say 6 (Fig. 132), then :—

$$h = \frac{s}{7}\{\tfrac{1}{2}(h_1 + h_7) + h_2 + h_3 + h_4 + h_5 + h_6\}.$$

Divide the base into any number of equal parts; add half the *sum of the end ordinates to the sum of all the others. Multiply the result by the common interval* s *to obtain the area: divide by the number of ordinates into which the figure is divided to obtain the mean ordinate. Mean ordinate × length = area of figure.*

It will perhaps be found very instructive to calculate by means of this rule the areas in the following examples, and compare the results obtained with the more accurate values as found by the application of Simpson's rules.

Simpson's Rule.—Simpson's rule provides a means of obtaining the area of any curvilinear area, $ABGD$ (Fig. 132), more accurately than by the method just described.

The base AB is divided into a number of equal parts (in this case 6). This ensures that the number of ordinates is an odd number, *i.e.* 3, 5, 7, 9, etc. Denoting as before the lengths of the ordinates AD, pm, sr, etc., by

$$h_1, h_2, h_3, \dots h_7.$$

Then if s denotes the common distance or space between the ordinates, we have

$$\text{Area of } ABGD = \frac{s}{3}\{h_1 + h_7 + 4(h_2 + h_4 + h_6) + 2(h_3 + h_5)\}$$

$$= \frac{s}{3}(A + 4B + 2C),$$

where A *denotes the sum of the first and last ordinate,*
where B „ „ *even ordinates,*
where C „ „ *odd ordinates.*

∴ *Add together the extreme ordinates, four times the sum of the even ordinates, and twice the sum of the odd ordinates (omitting the first and last). Multiply the result by one-third the common interval between two consecutive ordinates.*

The end ordinates at A and B may either one or both be zero, the curve commencing from the line AB (Fig. 132), but the rule still applies. In certain cases the figure may be bounded by two curved lines DG and EF, and the two straight lines DE and FG; the ordinates would then correspond to mn, rq, etc.; if the two curved boundaries are symmetrical about a line AB, it is only necessary to give the half-ordinates pm, rs, etc., each of these may be doubled, or, by substituting the values of the half-ordinates in Simpson's rule half the area is obtained. This, when doubled, is the area required.

Ex. 1. The half-ordinates of two curved lines *GD*, *EF* (Fig. 132), the common interval being 1·5 feet, are as follows: 2·3, 2·35, 2·46, 2·57, 2·42, 2·21, 2·10. Find the area.

Extreme ordinates, $2·3 + 2·10 = 4·4$.
Remaining odd ordinates, $2·46 + 2·42 = 4·88$.
Even ordinates, $2·35 + 2·57 + 2·21 = 7·13$.

$$\text{Area} = \frac{1·5}{3}(4·4 + 4 \times 7·13 + 2 \times 4·88) = 21·34 \text{ square feet.}$$

Hence area of *DEFG* $= 2 \times 21·34 = 42·68$ square feet.

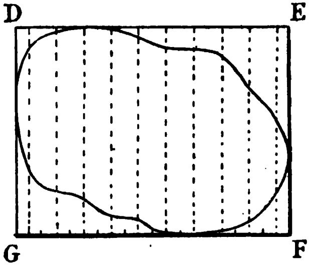


FIG. 133.

When the area is *not* symmetrical about a line; parallel lines such as *FG* and *ED* are drawn touching the curve, and *GD*, *FE* are also drawn touching the curve and perpendicular to *GF*.

As before, the base *GF* is divided into a number of equal parts and the ordinates of the curve measured; from these values, proceeding as before, the area can be obtained.

Ex. 2. The ordinates of a curve are 4, 5, 6, 7, 8, 12, and 14 feet; the common interval 2½ feet. Find the area of the curve.

$$\text{Area} = \frac{2\frac{1}{2}}{3}\{4 + 14 + 4(5 + 7 + 12) + 2(6 + 8)\}$$

$$= \frac{5}{6}(18 + 96 + 28) = \frac{5}{6} \times 142 = 116\frac{2}{3} \text{ sq. ft.}$$

Ex. 3. A force *P* acts along a line *AE*, the values of the force at *A* and at points 3, 6, 9, and 12 from *A* are respectively 50, 35, 28, 25, and 24 lbs. Draw a diagram of the work done by the force and calculate its amount in foot-pounds.

In Fig. 134 at the points along *AE*, ordinates (equal to the respective values of *P* at those points) are drawn as shown. If a fairly even curve *FG* be drawn through the points so obtained, the area *AFGE* represents work done.

If we add all the ordinates together the sum is 162; dividing this by 5, the number of ordinates, we obtain very roughly the mean ordinate to be $\frac{162}{5}$ lbs. This, multiplied by the length $AE = 12$ ft., gives the work in foot-pounds;

$$\therefore \frac{162}{5} \times 12 = 388\tfrac{4}{5}.$$

A much better result is obtained by Simpson's rule, thus:

$$\text{Area of curve} = \tfrac{3}{3}\{50 + 24 + 4(35 + 25) + 2(28)\}$$
$$= 74 + 240 + 56 = 370 \text{ foot-pounds.}$$

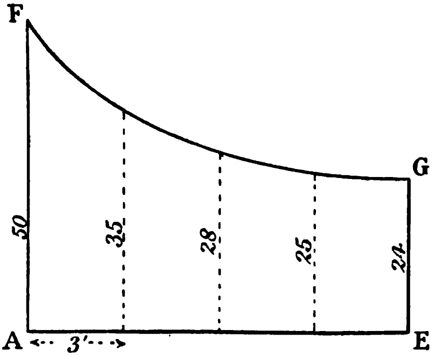


FIG. 134.

This example also furnishes a good instance of the superior accuracy of Simpson's rule. If we assume that no more than the three values at 50, 28, and 24 were given, as the common interval will now be 6 ft.

$$\text{Area} = \tfrac{6}{3}\{50 + 24 + 4(28) + 2 \times 0)\}$$
$$= 2(74 + 112) = 372 \text{ foot-pounds.}$$

Ex. 4. The length of the base is 16 inches, and the lengths of nine equidistant ordinates are as follows: 0, 1·5, 2·5, 3, 4·5, 5·5, 6·5, 7, 0; the common interval is 2 inches; find the area.

$$\text{Area} = \tfrac{2}{3}\{(0 + 0) + 4(1{\cdot}5 + 3 + 5{\cdot}5 + 7) + 2(2{\cdot}5 + 4{\cdot}5 + 6{\cdot}5)\}$$
$$= \tfrac{2}{3}(68 + 27) = \tfrac{2}{3} \times 95 = 61\tfrac{2}{3} \text{ sq. inches.}$$

Average Cross-Section.—The volume of a solid is obtained by multiplying together its average cross-section and its length. Simpson's rule may be applied with advantage to obtain the average cross-section, when the cross-sections of an irregular body at given intervals are known. In the rule it is only necessary to substitute the words 'area' for 'ordinate' and 'volume' for 'area,' if h denote the mean height or average cross-section.

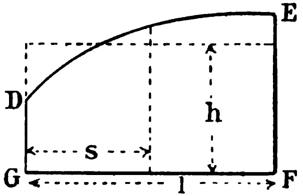


FIG. 135.

As the area is given by

$$A=\frac{s}{3}(A+4B+2C),$$

and the length AB are known, the value of h can easily be obtained.

An important practical case occurs when only three ordinates (Fig. 135) are given.

In this case C is 0; $\therefore$ area of $GFED=\frac{s}{3}(A+4B)$;

but $2s\times h=l\times h$ is also the area of $GFED$;

$$\therefore\quad 2s\times h=\frac{s}{3}(A+4B);$$

$$\therefore\quad h=\frac{1}{6}(A+4B),$$

or

$$\text{average section}=\frac{1}{6}\left\{\begin{matrix}\textbf{sum of first and last sections}\\ \textbf{+4 times the middle section}\end{matrix}\right\}.$$

To avoid, as far as possible, mistakes finding their way into the work, other methods consisting of the arrangement of the given numbers into vertical rows are used. This can be best shown by an example.

Ex. 4. Find the cubical contents of a log of timber 30 feet long, the cross-sectional areas at intervals of 5 feet being respectively 7·5, 5·08, 3·54, 2·52, 1·86, 1·34, 0·92 square feet. Find also what the volume would be if only the cross-sectional areas of the two ends and the middle were given.

(1) Ordinate,	7·5	5·08	3·54	2·52	1·86	1·34	0·92
(2) Simpson's Multiplier,	1	4	2	4	2	4	1
(3) Product,	7·5	20·32	7·08	10·08	3·72	5·36	·92

Adding the numbers in the last row (3) together, the sum is found to be 54·98.

As the common interval is 5 feet,

$$\text{volume} = \frac{54{\cdot}98 \times 5}{3} = 91{\cdot}6 \text{ cub. ft.}$$

When only three ordinates are given,

then $A = 7{\cdot}5 + {\cdot}92 = 8{\cdot}42$;

also $4B = 2{\cdot}52 \times 4 = 10{\cdot}08$;

$$\therefore \ A + 4B = 18{\cdot}5.$$

The common interval is now 15 feet;

$$\therefore \ \text{volume} = \frac{18{\cdot}5 \times 15}{3} = 92{\cdot}5 \text{ cub. ft.}$$

Or, we could find the mean or average cross-section, and multiplying by the length obtain the volume.

Thus, $\text{average section} = \frac{1}{6}(8{\cdot}42 + 4 \times 2{\cdot}52) = \frac{18{\cdot}5}{6}$;

$$\therefore \ \text{volume} = \frac{18{\cdot}5}{6} \times 30 = 92{\cdot}5, \text{ as before.}$$

Simpson's Second Rule.—When an even number of ordinates (Fig. 136) is used Simpson's Second Rule may be employed.

$$\therefore \ \text{Area of } DEFG = \frac{3s}{8}(h_1 + 3h_2 + 3h_3 + h_4),$$

if h denote mean height. Then $h \times DE = $ area;

$$\therefore \ 3s \times h = \frac{3s}{8}(h_1 + 3h_2 + 3h_3 + h_4);$$

$$\therefore \qquad h = \frac{1}{8}(h_1 + 3h_2 + 3h_3 + h_4).$$

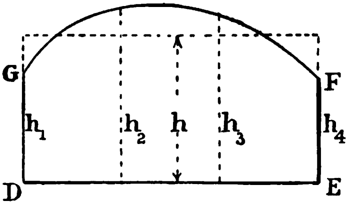


FIG. 136.

The first rule is the best known and most commonly used, and is usually known simply as *Simpson's Rule.* The following examples are all to be solved by means of the first rule. It will, however, be found useful to calculate the areas and volumes in a few cases by the second rule, and compare the results with those obtained by using the first rule.

Simpson's Rule may be applied to find the average force acting on a body, when the actual force acting at the various distances are known. This may be shown by the following example:

Ex. 5. A body of mass 200 lbs. is acted upon by a force F; the force in lbs.-weight is as follows:

Distance passed through by the body in feet,	0	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9	1·0
Force F,	25	26	26	24	21·5	20	17	12·5	7	3	0

Find (*a*) the average force acting on the body during the first foot;

(*b*) the work done on the body during the first half of a foot;

(*c*) if the velocity is 10 feet per second when the distance is 0, find the velocity when the distance is half a foot.

(*a*) Applying Simpson's Rule:

$A=25,\quad C=71{\cdot}5;\quad \therefore\ 2C=143,\ B=85{\cdot}5,\ \text{and}\ 4B=342.$

As the common interval is $\frac{1}{10}$ foot;

$$\therefore\ \text{area}=\tfrac{1}{30}(25+143+342)=\tfrac{510}{30}$$
$$=17 \text{ square feet.}$$

As the length is 1 foot, the average pressure (or height of equivalent rectangle) is 17;

$$\therefore\ \text{average pressure}=17 \text{ lbs.}$$

The answers to (*b*) and (*c*) are easily obtained.

Ans. 12; 4·37.

Centre of Area or Gravity.—Simpson's rule affords a ready means of obtaining the centre of area, or as it is usually called, the centre of gravity of an irregular area. In the majority of cases the figure is symmetrical about a line, and the position of the centre of area along this line has to be determined.

To effect this, denoting the distance of the centre of area from one end by $\bar{x}$ we have:

$\bar{x}\times$(total area)=(area of each strip)$\times$(distance of each strip from one end). See p. 315.

We may apply the rule to the following example:

Ex. 1. The half ordinates in feet of a symmetrical area (Fig. 137) are ·0, 8·0, 9·6, 10·8, 11·0, 10·2, 8·6, 5·0, 0; find the area, and the position of the centre of area, common interval, 4 feet. If the figure were drawn to scale, the ordinates shown by dotted lines could be measured, and each multiplied by 4 would denote the area of each strip approximately. If each area so obtained were multiplied by its distance from some point such as *E*, then by adding all the products together and dividing by the total area the position of the centre of area is determined, or we may tabulate as before.

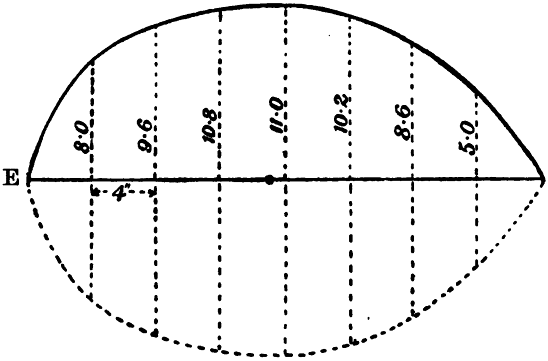


FIG. 137.

(1) Ordinate,	0	8·0	9·6	10·8	11·0	10·2	8·6	5·0	0
(2) Simpson's Multiplier,	1	4	2	4	2	4	2	4	1
(3) Product,	0	32	19·2	43·2	22·0	40·8	17·2	20·0	0
(4) Moment,	0	32·0	38·4	129·6	88·0	204·0	103·2	140	0

The sum of the numbers in (3) amount to 194·4 and in column (4) to 735·2;

$$\text{area} = 194\cdot4 \times \tfrac{4}{3} = 259\cdot2.$$

To find *centre of area.*—Each product in column (3) is multiplied by its distance from *E*. Thus $32\cdot0\times1\times4$, $19\cdot2\times2\times4$, $43\cdot2\times3\times4$.

As the common multiplier 4 occurs in each product, it is best to obtain the numbers as shown, and finally multiply 735·2 by 4, giving a product of 2940·8 ;

$$\therefore\quad \bar{x}=\frac{2940\cdot8}{194\cdot4}=15\cdot13,$$

or the distance of centre of area from $E=15\cdot13$ ft.

EXERCISES. LXVII.

1. The lengths of seven equidistant ordinates of an irregular piece of ground are respectively 15, 19, 20, 23, 25, 30, and 33 feet, and the length of the base is 72 feet. Find the area.

2. Draw a circle of 12 inches diameter, and find by Simpson's rule the area of a quadrant, or a semicircle. Compare the result with that obtained by calculating the area of the circle and dividing by 4, or 2.

3. Find the area enclosed by a curved line and three straight lines, the ordinates being 8, 10, 12, 14, and 15, and the length of the base 24.

4. The ordinates of a curve are in feet 1·1, 1·075, 1·125, 1·15, 1·075, ·95, ·975, 1·1, 1·225, 1·325, 1·225, the common interval 4 feet, find the area.

5. The length of the perpendiculars let fall from points in an irregularly curved line of fence upon a straight line of 5·86 chains, at equal distances from each other, are found to be 93, 84, 72, 68, 43, 54, 37, 29, and 23 links. Find the area enclosed between the curved line, the extreme ordinates, and the base line.

6. The half-ordinates in feet of the midship section of a vessel are 12·5, 12·8, 12·9, 13, 13, 12·8, 12·4, 11·8, 10·4, 6·8, and ·5 respectively, and the common interval is 2 feet. Find the area of the whole section.

7. The half-ordinates of a deck of a vessel are ·5, 7·6, 16·7, 24·4, 28·9, 30·3, 29·9, 27·3, 22·3, 14·9, and 6 feet respectively, the common interval between the half-ordinates being 25 feet. In the deck are two rectangular hatchways, one being 12 feet by 4 feet, and the other 10 feet and 3½ feet. Find the area of the deck excluding the hatchways.

8. The half-ordinates of a water plane are 15 feet apart, and are 1·9, 6·6, 11, 14·5, 17·4, 19·4, 20·5, 20·8, 20·3, 18·8, 15·8, 10·6, and 2·6 feet respectively. Find the area of the plane in square feet, and the position of its centre of area.

9. The semi-ordinates of the boundary of the deck of a vessel in feet are respectively ·1, ·5, 11·6, 15·4, 16·8, 17, 16·9, 16·4, 14·5, 9·4, and ·1; the semi-ordinates being 11 feet apart. Find the area of the deck in square yards.

10. The semi-ordinates of the load-water plane of a vessel are ·2, 3·6, 7·4, 10, 11, 10·7, 9·3, 6·5, and 2 feet respectively, and they are 15 feet apart. What is the area of the load-water plane.

11. The half-ordinates of the midship section of a vessel are 12·7, 12·8, 12·9, 12·9, 12·9, 12·8, 12·5, 11·9, 10·4, 5·9, and 1·4 feet respectively, and the common interval between them is 18 inches. Find the area of the section in square feet.

12. If in Ex. 6 the common interval was 6 instead of 15 feet, find the area.

Other Methods of finding the area of an irregular figure, instead of the foregoing, are by means of squared paper, by weighing, and by using a planimeter.

Squared Paper.—Draw the figure on squared paper, estimate the number of whole squares within the boundary and the fractional parts of those cut by it, add the latter to the former, the sum will approximately be the area required.

By Weighing.—Draw the figure to scale on thick paper or cardboard of uniform thickness, cut it out carefully, also cut out a rectangular piece from the same sheet, find the weight of the rectangular piece, and hence deduce the weight of a square inch. Then, knowing the weight of the irregular figure, its area can be found.

By the Planimeter.—The planimeter is an instrument for estimating readily and accurately the areas of irregular figures.

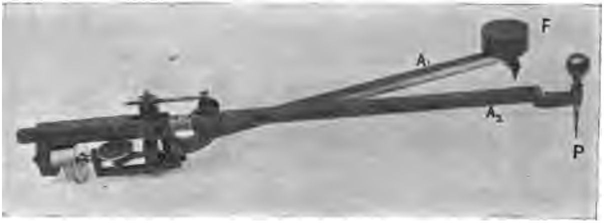


FIG. 138.—The Planimeter.

There are many forms of the instrument, to which various names are applied, but they are mostly modifications of the *Amsler planimeter.*

The general appearance of the instrument is shown in Figs. 138 and 139 and consists of two arms A_1 and A_2 pivoted together at a point O.

The arm OA_1 is fixed at some convenient point by the point F, being held in position by means of the weight shown. The other arm OA_2 carries a tracing point P. This is passed round the outline of the curve or figure the area of which is required. The arm OA_2 carries a wheel W, the rim of which is usually divided into 100 equal parts.

When the instrument is in use the rim of the wheel rests on the paper, and as the point P is carried round the outline of the figure, gives motion, by means of the spindle S, to the dial D.

One rotation of the wheel corresponds to one-tenth of a revolution of the dial or one-tenth of a unit.

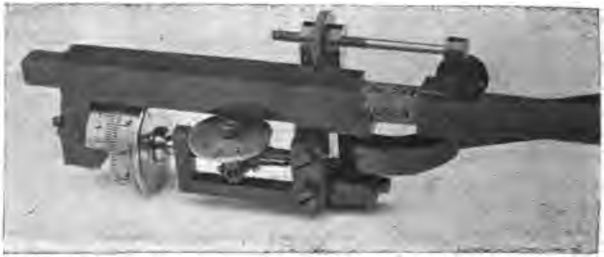


FIG. 139.—Planimeter showing the vernier arrangement.

A vernier shown in Fig. 139 is fixed to the frame, and as a distance equal to 9 scale divisions on the rim of the wheel is divided into 10 on the vernier, a reading is readily taken to three decimal places.

Thus, to obtain the reading indicated in Fig. 139 we note the division 9 on dial, 5 on the wheel, and referring to the vernier the next figure is 3 and a fraction. To obtain the fractional part we must obtain coincidence of the scale on the vernier with that of the scale on the rim. This occurs at the 6th division on the vernier. Hence the reading is 9·536.

Summary.

The area of an irregular figure can be obtained thus:

Divide the figure into a number of equal spaces, and erect ordinates at the centre of each space:

(1) Add the lengths of all the ordinates together, and, to obtain the mean ordinate, divide the sum by the number of ordinates.

(2) **Trapezoidal Rule.**—Add half the sum of the end ordinates to the sum of all the others, and divide this by the number of ordinates to obtain the mean ordinate.

In each of the above the area = length × mean ordinate.

(3) **Simpson's Rule.**—Add together the extreme ordinates, four times the sum of the even ordinates, and twice the sum of the odd ordinates. Multiply the result by one-third the common interval.

Average Cross-Section. $= \frac{1}{6}$ (sum of first and last sections $+ 4$ times the middle section).

By use of Squared Paper.—Draw the figure on squared paper, estimate the number of whole squares within the boundary, and the fractional parts cut by it, and add the two together.

Area by Weighing.—Draw the figure on a sheet of paper of uniform thickness, such as a sheet of cardboard: cut out the figure as carefully as possible: also cut out of the same piece of cardboard a rectangular piece, weigh the latter; hence deduce the weight of a square inch, from this knowing the weight of the irregular figure its area can be found.

By the **Planimeter.**—An area is readily obtained without calculation by means of a planimeter.

CHAPTER XVII.

MENSURATION. SOLIDS. AREAS, VOLUMES AND WEIGHTS OF PRISMS (RIGHT AND OBLIQUE). CYLINDER, HOLLOW CYLINDER, PYRAMID, CONE, FRUSTUM OF PYRAMID, FRUSTUM OF CONE. GULDIN'S THEOREM. SURFACE AND VOLUME OF SPHERE, CYLINDER, RING, FLY-WHEEL.

A solid figure or **solid** is a figure having the three dimensions of length, breadth and thickness. When the surfaces bounding a solid are plane, they are called faces, and the edges of the solid are the lines of intersection of the planes forming its faces.

What are called the regular solids are five in number, viz., the *cube*, *tetrahedron*, *octahedron*, *dodecahedron*, and *icosahedron*.

The *cube* is a solid having six equal square faces.

The *tetrahedron* has four faces all equilateral triangles.

The *octahedron* has eight faces all equilateral triangles.

The *dodecahedron* has twelve faces all pentagons.

The *icosahedron* has twenty faces all equilateral triangles.

Models of these five solids can be made from cardboard by drawing the developments of the solids to any convenient size.

Cylinder.—If a rectangle *ABCD* (Fig. 140) be made to revolve about one side *AB*, as an axis, it will trace out a cylinder. Or, a cylinder is traced by a straight line always moving parallel to itself round the boundary of any curve, called the *guiding curve*.

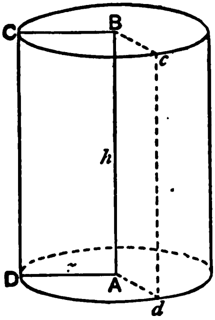


FIG. 140.

Pyramid.—If one end of the line always passes through a fixed point, and the other end be made to move round the boundary of any curve, a *pyramid* is traced out.

Cone.—If the curve be a circle and the fixed point is in the line passing through the centre of the circle, and at right angles to its plane, a *right cone* is obtained; an *oblique cone* results when the fixed point is not in the line at right angles to the plane of the base.

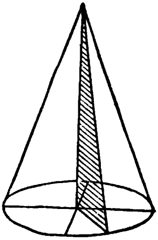


FIG. 141.

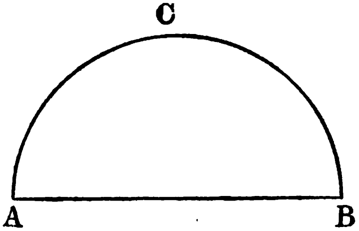


FIG. 142.

Sphere.—If a semi-circle *ACB* (Fig. 142) revolve about a diameter *AB*, the surface generated is a *sphere.*

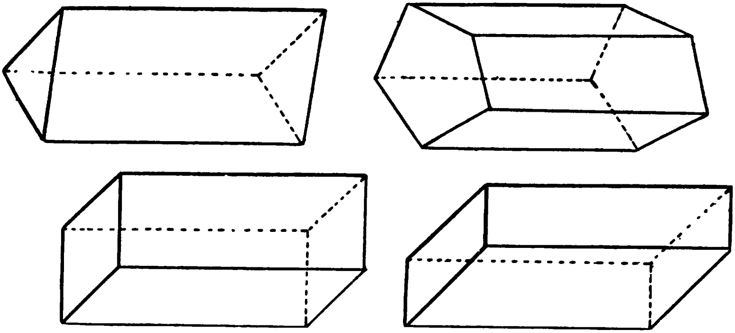


FIG. 143.

Prism.—When the line remains parallel to itself and is made to pass round the boundary of any rectilinear polygon, the solid formed is called a *prism.*

The ends of a prism and the base of a pyramid may be polygons of any number of sides, *i.e.*, triangular, rectangular, pentagonal, etc.

A prism is called *triangular*, *rectangular*, *square*, *pentagonal*, *hexagonal*, etc., (Fig. 143) according as the end or base is one or other of these polygons.

A prism which has six faces all parallelograms is called a *parallelopiped.*

A right prism has its side faces perpendicular to its ends; other prisms are called **oblique.**

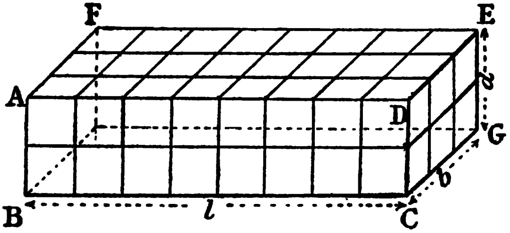


FIG. 144.—Volume of a right prism or parallelopiped.

In (Fig. 144) a right prism, the ends of which are rectangles, is shown; to find its volume it is necessary to find the area of one end *DCGE*, and multiply by the length *BC.* Let l, b and d denote the length, breadth, and depth respectively.

Then area of one end $= b \times d$.

And *volume of prism* $= b \times d \times l$.

As $b \times l =$ area of base; volume $=$ *area of base* × *altitude.*

Ex. 1. If the length be 8 ft., the depth 2 ft., and breadth 3 ft.

Area of one end $= 3 \times 2 = 6$ sq. ft.;

$\therefore$ volume $= 6 \times 8 = 48$ cub. ft.

If, as in Fig. 144, the length *BC* be divided into 8 equal parts, the breadth into 3, and the depth into 2. Then it will be seen that there are 6 square units in the end *DCGE*, and 8 rows, hence the volume is $8 \times 6 = 48$ cubic feet.

Total Surface of a Right Prism.—The total surface is, from Fig. 144, seen to be twice the area of the face *ABCD*, and twice the area of *ADEF*, together with the area of the two ends.

∴ Surface $=2(ld+bl+bd)$, or *perimeter of base multiplied by altitude together with areas of the two ends.*

Volume of a Cube.—When the length, breadth, and depth are each equal the solid is called a *cube*, and if a denote the length of edge of the solid the area of the base is a^2 and the volume is $\mathbf{a}^3$.

Ex. 2. Let the length of the edge of the cube be 4 inches (Fig. 145). It is seen by dividing two adjacent sides (or edges), as EB and BC, each into 4 equal parts and drawing through the points of division lines parallel to EB and BC that one end $BCDE$ contains 16 square inches, and as there are four such rows, the volume is $4 \times 16 = 64$ cubic inches.

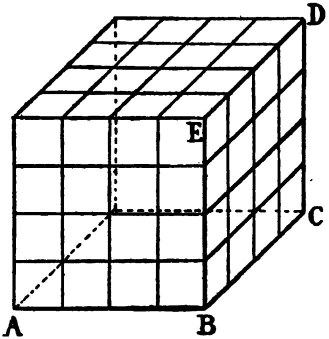


Fig. 145.—Volume of a cube.

Volume of an Oblique Prism.—The volumes of all parallelopipeds having the same or equal bases and the same altitudes are equal.

In Fig. 146 an oblique prism $ABCDEF$ is shown. By drawing CN and DH perpendicular to DC, and NP parallel to BF, wedge-shaped pieces are obtained. Assuming the wedge-shaped piece $CNPFB$ transferred to the left, as indicated, the oblique prism becomes a right prism on a rectangular base.

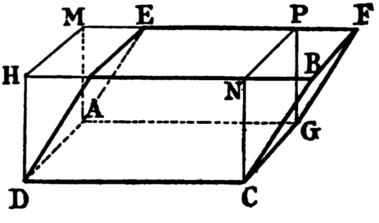


Fig. 146.—Volume of an oblique prism.

Thus *the volume of an oblique prism or parallelopiped is equal to the area of the base multiplied by its altitude.*

Perhaps the best and the easiest method is to build up a rectangular prism from a number of thin rectangles (of millboard, cardboard, or thin wood). By shearing the solid so constructed, as in Fig. 147, any degree of obliquity can be obtained, and the height, which is obviously the sum of the thicknesses of the rectangles, remains the same. Hence the volume of the solid is unaltered.

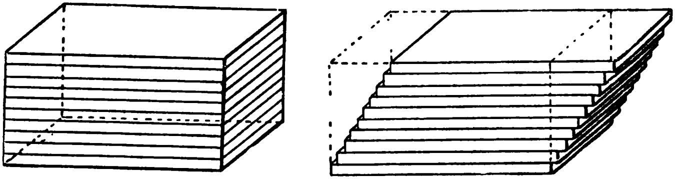


FIG. 147.—Model to illustrate a right and an oblique prism.

When the volume of any solid is known, then the weight can be obtained by multiplying the volume by the weight of unit volume. The weights of unit volumes of the materials usually used are given in Table III., p. 371.

Thus, a rectangular piece of wrought iron, of thickness $\frac{1}{2}$ inch, 8 inches wide, and 12 inches in length, would weigh

$$8 \times \tfrac{1}{2} \times 12 \times \cdot 28 = 13 \cdot 44 \text{ lbs.}$$

EXERCISES. LXVIII.

1. Find the volume of a beam 10 ft. 6 in. long, 4 in. wide, and 2 in. thick.

2. Find the solid content of a log of timber 10 yds. 2 ft. 7 in. long, 2 ft. 11 in. broad, and 2 ft. 5 in. thick.

3. The length of a ditch, horizontal at top and bottom, is 100 yds., its depth 12 ft., width at top 12 ft., and at bottom 8 ft. Find its content.

4. A ditch is 5000 ft. long, 9 ft. deep, 14 ft. broad at top, and 11 ft. broad at the bottom ; how many cubic feet of water will fill it? If half that quantity of water is supplied, how high will it rise?

5. How many cubic feet of water are contained in a ditch, shaped like the frustum of a wedge, 120 yds. long, 6 ft. deep, 10 yds. broad at the top, and 4 yds. at the bottom.

6. A rectangular tank is 14 ft. long and 10 ft. wide; what must be its depth that it may just contain 31¼ tons of water, a cubic foot of water weighing 62½ lbs.?

7. The cross section of a canal is a trapezoid; width at water-level 47 ft., and at the bottom of the canal 29 ft., depth 11 ft. Find its cross-sectional area.

8. Find the weight of a uniform beam of oak 10 ft. long, 15 in. deep, and 10 in. wide. The specific gravity of oak = ·934.

9. A reservoir measures 80 yds. square at the base, and the sides slope outwards at the rate of 1½ ft. horizontal to 1 ft. vertical. Find the content in cubic yards and in gallons when filled to a depth of 10′ 6″.

10. A block of stone measures 4′ 5″ × 2′ 9″ × 1′ × 11″. Calculate the solid content and find the weight, assuming the specific gravity to be 2·78.

11. A seam of coal is 3′ 10″ in thickness and the specific gravity is 1·285. How many tons per acre will it produce, if 9 per cent. be deducted for working?

12. The bottom of a water tank measures 7′ in length and 3′ 4″ in width. When the tank contains 900 gallons of water, what will be the depth of the water, and what will be the pressure on the bottom?

13. The total depth of a cast-iron beam is 13 in. The top flange is 6″ by $\frac{7}{8}''$, the bottom flange 9″ × $1\frac{1}{8}''$, and the web is $\frac{5}{8}''$ thick. Find the weight of the beam per foot length.

14. How long will it take to empty a reservoir of water 15 ft. long, 10 ft. 2 in. wide, and 10 ft. deep, at the rate of 10 gallons a minute, a gallon containing $277\frac{3}{11}$ cub. in.?

15. (*a*) How many cubic feet of lead, $\frac{1}{16}$ in. thick, will be required to cover the sides and bottom of a cistern 10 ft. long, 6 ft. 6 in. wide, and 7 ft. deep? (*b*) What weight of water will the cistern hold, a cubic foot of water weighing 1000 ounces?

16. Find the cubical content of a cistern 36 ft. 8 in. long, 7 ft. 11 in. broad, and 6 ft. 5 in. high, giving the answer in cubic yards.

17. Find the volume of water which will fill a tank 5 ft. long, 4 ft. 7 in. wide, and 2 ft. 4½ in. deep.

18. Find the volume of a room 14 ft. 7 in. long, 12 ft. 5 in. wide, and 11 ft. 3 in. high.

19. A stack of 125,000 bricks is 41 ft. 8 in. long, 16 ft. 8 in. wide. and 14 ft. 7 in. high; if each brick is 9 in. long and 3 in. thick, find its breadth. Prove your answer.

20. A room is 20 ft. long, 18 ft. 6 in. broad, and 11 ft. high; it has a door 4 ft. by 7 ft., and a window 5 ft. by 8 ft. Find the cost of papering the room at 9d. a square yard.

21. Find the cost of painting the 4 walls of a room at 10d. a square yard, the length of the room being 18 ft. $10\frac{1}{4}$ in., breadth 17 ft. $1\frac{3}{4}$ in., and height 12 ft. 3 in.

22. A block of stone contains 5 cub. ft. 24 cub. in.; its breadth and thickness are each 2 feet; find its length.

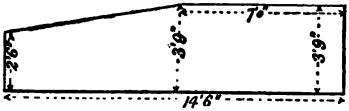


FIG. 148.

23. A steel plate, $\frac{5}{8}$ in. in thickness, is of the form and dimensions shown (Fig. 148). Find its volume and weight.

24. A steel plate, $\frac{5}{8}$ in. thick, is of the form and dimensions shown (Fig. 149). Find its volume and weight.

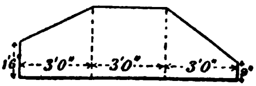


FIG. 149.

25. A Dantzic oak deck plank is 25 ft. 6 in. long, and $4\frac{1}{2}$ in. thick. It is 8 in. wide at one end and tapers gradually to 6 in. at the other end. Find its volume and weight.

26. A steel plate is 12 ft. 9 in. long, $5\frac{1}{2}$ in. broad, and its thickness tapers from 10 in. at its upper edge to 6 in. at its lower edge. Find its volume in cubic feet, and the weight in tons.

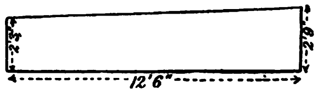


FIG. 150.

27. An iron plate of the form and dimensions shown (Fig. 150) is of uniform thickness of $\frac{3}{4}$ in., what is its weight?

28. The flanges of a rolled iron joist are $4'' \times 1''$, the web $8'' \times 1''$. If the joist be 12 ft. in length, find its weight.

29. The flanges of a cast-iron girder are $3'' \times 1''$ and $9'' \times 1''$ respectively, the web 1″ thick and 10″ long. If the length of the girder is 10 ft., find its weight.

Cylinder.—It has been seen that the volume of a prism is equal to the area of the base multiplied by the altitude (or height).

In the case of a cylinder the base is a circle.

If r denote the radius of the base and h the altitude or height of the cylinder (Fig. 134).

$$\textit{Area of base} = \pi r^2. \quad \therefore \textit{Volume of cylinder} = \pi r^2 \times h.$$

Lateral Surface.—The surface of a cylinder consists of two parts, the curved surface of the sides, called the *lateral surface*, and the two ends which are plane circles.

If the cylinder were covered by a piece of thin paper this when unrolled would form a rectangle of height h and base $2\pi r$. Thus if the lateral surface of a cylinder be conceived as unrolled and laid flat, it will form a rectangle of area $2\pi r \times h$ (Fig. 151).

$\therefore$ Lateral surface of cylinder $= 2\pi rh$.

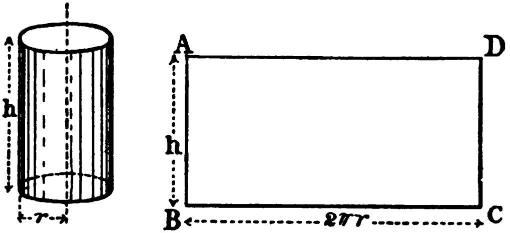


FIG. 151.—Surface of a cylinder.

To obtain the whole surface the areas of the two ends must be added to this.

$$\therefore \text{Total surface of cylinder} = 2\pi rh + 2\pi r^2.$$
$$= 2\pi r(h + r).$$

Experimental Method.—The cylinder can be built up from a number of thin wood or cardboard discs (Fig. 152). The volume of one disc can be ascertained. This multiplied by the number of discs will give the volume of the cylinder.

Ex. 1. If 20 discs are used, each 3 inches diameter and $\frac{1}{4}$ inch in thickness,

$$\text{The area of base of one disc} = \pi \times \left(\tfrac{3}{2}\right)^2 = 7 \text{ sq. in.},$$
$$\text{and volume} = 7 \times \tfrac{1}{4} = \tfrac{7}{4} \text{ cub. in.};$$
$$\therefore \text{ volume of cylinder} = 20 \times \tfrac{7}{4} = 35 \text{ cubic inches.}$$

Oblique Cylinder.—By using a simple arrangement of the kind suggested, as in the case of an oblique prism, by shearing the solid (Fig. 152) any degree of obliquity can be obtained, the volume remaining unaltered.

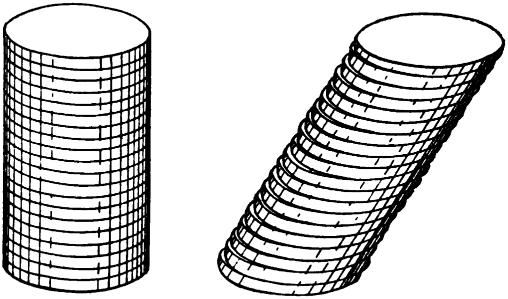


FIG. 152.—Model to illustrate a right and an oblique cylinder.

Hence *volume of oblique cylinder = area of base × altitude*

$$= \pi r^2 \times h$$

Section of a Cylinder.—When a cylinder is cut by a plane (Fig. 153) not perpendicular, or parallel to the axis, the section is an ellipse.

The volume of the solid is the product of the area of base and altitude.

The altitude h is the perpendicular distance from the centre of the base AC to the centre of the ellipse BD.

If points B and D are the lowest and highest points respectively; then the height h is half the sum of AB and CD.

For, if we assume the cylinder to be cut by a plane parallel to the base and passing through the centre of the ellipse, a wedge-shaped portion is obtained which may be transposed as shown.

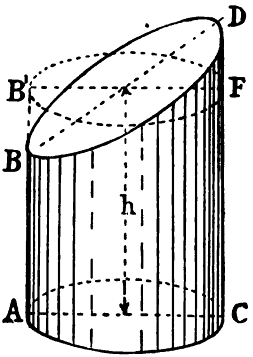


FIG. 153.—Frustum of a cylinder.

Frustum of a Cylinder.—The part of a cylinder cut off by a *plane not parallel* to the base is called the *frustum of a cylinder*.

In Fig. 153 the frustum of a cylinder is shown by the figure $BACD$. Let $BA=h_1$ and $CD=h_2$.

Volume of Frustum of Cylinder.—This, as will be seen from Fig. 153, is the area of the base multiplied by the altitude h;

$$\therefore \quad \text{volume}=\pi r^2\times h=\pi r^2\times\left(\frac{h_1+h_2}{2}\right).$$

Cross-Section.—The term cross-section should be clearly understood; as already explained, a section of a cylinder by any plane perpendicular to the axis of the cylinder is a circle; any oblique section gives an ellipse. Hence, the term *area of cross-section* is used to indicate the area of a section (a circle) at right angles to the axis.

EXERCISES. LXIX.

1. Find the surface and solid content of a cylinder 80 ft. long, the radius of the base being 8 ft.

2. Find the number of gallons of water contained in a cylinder, the radius of the base of which is 18 inches, the height 20 feet; 1 gal. = 277·27 cub. in.

3. In a force pump, dia. of plunger = 17″. If the height to delivery pipe be 220 ft., find weight of column of water lifted.

4. Exclusive of the crown what are the cubical contents of a gasometer 100 ft. in diameter by 20 ft. deep?

5. A cylindrical tank 6 ft. in diameter and 10 ft. deep is filled with water; find the pressure on the base.

6. How deep must the water be in a rectangular reservoir whose length is 160 ft. and breadth 43 ft., in order that the volume of water may be the same as in a cylindrical one in which the depth of the water is 12 ft. and the diameter of the base 80 ft.?

7. A mass of gold can be beaten into gold leaf extending over 10,100 square yds. and $\frac{1}{39000}$ in. thick. Find to the nearest yard what length of wire $\frac{1}{19}$ in. thick could be made out of the same mass of gold.

8. A cylindrical gasometer 200 ft. internal diameter is required to hold 5 million cubic feet of gas; find its height.

   Also find its weight neglecting the joints, if the material is wrought iron $\frac{3}{8}$ in. thickness.

9. An imperial bushel measure is a hollow cylinder, the internal diameter of which is twice the depth. Find the diameter and the depth (an imperial bushel contains 8 gallons).

10. The internal dimensions of a rectangular tank are, length, $4\frac{1}{2}$ ft.; width, 5 ft.; it is filled with water to a height of 12 in. If the water be transferred to a cylinder of radius 12 in., find the height to which it will rise.

11. A rectangular bar of silver, whose dimensions are 9 in., $4\frac{1}{4}$ in., and $2\frac{7}{8}$ in., is drawn out into wire 270 ft. long. Show that its thickness is less than $\frac{3}{10}$ in.

12. Find the weight of a piece of steel shafting 3 in. diameter and 20 ft. in length.

13. The safety valve of a boiler is loaded with 10 cast-iron weights 10 in. diameter and 2 in. thick. Find the total weight on the valve.

If the valve and its appendages weigh 40 lbs. find the necessary pressure of the steam to lift the valve when its diameter is 3 in.

14. There are in a surface condenser 1000 brass tubes, each 6 ft. in length and 1 in. outside diameter. What amount of cooling surface will such a condenser provide?

Hollow Cylinder.—The volume is as before, area of base multiplied by the altitude.

If R and r denote the radii of the outer and inner circles respectively, D and d the corresponding diameters,

$$\text{area of base} = \pi R^2 - \pi r^2 = \pi(R^2 - r^2)$$
$$\text{and volume} = \pi(R^2 - r^2)h$$
$$= {}\cdot 7854(D^2 - d^2)h.$$

Ex. 1. Find the weight of a cast-iron roller internal diameter $22\frac{1}{2}$ inches, the thickness of the metal $\frac{3}{4}$ inch, and the altitude 3 feet.

$$\text{The outer diameter} = 22\tfrac{1}{2} + 2 \times \tfrac{3}{4} \text{ inches} = 24 \text{ inches.}$$
$$\text{Area of base} = {}\cdot 7854(2^2 - 1\cdot 875^2);$$
$$\therefore \text{ volume} = {}\cdot 7854(2^2 - 1\cdot 875^2) \times 3 = 1\cdot 5217 \text{ cub. ft.}$$

From Table III., p. 371, the weight of a cubic foot of cast iron is 450 lbs.;

$$\therefore \text{ weight of roller} = 1\cdot 5217 \times 450 \text{ lbs.} = 684\cdot 76 \text{ lbs.}$$

Ex. 2. Find the volume and weight of a cast-iron cylinder, radius of base 1 ft. 9 in., height 4 ft. 6 in.

$$\text{Area of base} = (1\cdot 75)^2 \times {}\cdot 7854,$$
$$\text{volume} = (1\cdot 75)^2 \times {}\cdot 7854 \times 4\cdot 5 = 10\cdot 82 \text{ cub. ft.},$$
$$\text{weight} = 10\cdot 82 \times 450 = 4869 \text{ lbs.}$$

EXERCISES. LXX.

1. The rim of a fly-wheel of cast iron is 7 feet external and 5 feet internal diameter. Find its weight, the mean thickness being 1½ inches.

2. The cross-section of the rim of a cast-iron fly-wheel is a rectangle 8 inches wide, 2 inches mean thickness. If the mean diameter of the rim be 6 feet find the weight of the rim.

3. A steel shaft is 10 inches diameter and 20 feet in length. Find the external diameter of a hollow shaft 4 inches internal diameter and same length to give the same weight.

4. A portion of a cylindrical steel stern shaft casing is 12¾ feet in length, 1¼ inches thick, and its external diameter is 14 inches. Find its volume and weight.

5. A rectangular steel deck plate is 14 feet 3 inches long, 3 feet 3½ inches wide, and ½ inch thick. A circular piece 13 inches diameter is cut out of the centre of the plate. What is the weight of the plate?

6. A solid pillar of iron of circular section is 6 feet 10 inches long and 2¼ inch diameter. What is its volume and weight?

7. A solid pillar of iron is of uniform circular section. It is 7 feet 3 inches long and 2⅜ inches diameter. What is its weight?

8. The external diameter of a hollow steel shaft is 18 inches, and its internal diameter 10 inches. Calculate the weight of a 20 feet length of this shafting.

9. The dimensions of a locomotive crank axle is given in Fig. 154; find its weight.

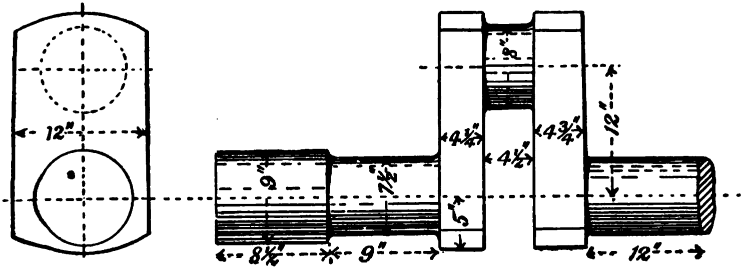


Fig. 154.

10. A brass lining for a cylinder is 4 feet internal diameter, 4½ feet long, and 1 inch thick. Find its weight, 1 cubic foot weighs 515 lbs.

11. A boiler contains 510 smoke tubes each $2\frac{5}{8}$ inches internal diameter. Find the total area for draught through them.

12. The smoke tubes in a boiler are 7 feet long between the tube plates, the tube plates are $\frac{3}{4}$ inch thick, and the tubes project $\frac{3}{16}$ inch from each tube plate. The tubes are $2\frac{3}{4}$ inches external diameter, and $2\frac{1}{8}$ inches internal diameter. Find the weight of one tube.

13. In Fig. 155 a sketch of a cast-iron pump barrel is shown. Calculate its weight, assuming the weight of the two flanges to be equal to one foot length of barrel.

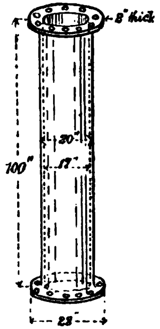


FIG. 155.

14. In Fig. 156 the dimensions of a cast-iron muff coupling are given. If the shaft is wrought iron, find the weight of the shaft and also the weight of the coupling.

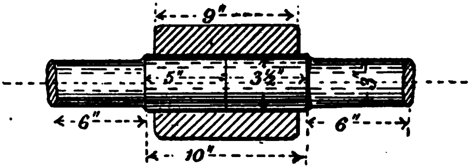


FIG. 156.

15. A cubical mass of brass, whose edge is $7\frac{1}{2}$ inches, is drawn out into wire $\frac{1}{11}$ inch in diameter. Find its length.

16. A rectangular bar of brass, whose length is $13\frac{1}{2}$ inches, and whose cross-section is a square (side $3\frac{3}{4}$ inches) is drawn out into wire 196 yards long. Find the weight of 93 feet of this wire.

17. Find the weight of a solid cylindro-conical steel projectile, the whole length being 13 inches, the length of the conical part 4 inches, and the diameter of the circular end being $3\frac{1}{2}$ inches.

18. The cross-section of a brick subway 30 feet long is a rectangle surmounted by a semicircle, the total height and breadth are 9 feet and 5 feet 6 inches, and the thickness of the brickwork is $4\frac{1}{2}$ inches. Find the weight if a brick containing $\frac{6 \cdot 3}{100}$ of a cubic foot weighs 7 lbs.

19. A circular brick chimney is 13 feet outside diameter at the bottom and 10 feet at the top. If the mean internal diameter be 8 feet and the height 140 feet find its weight. (A cubic foot of brickwork weighing 112 lbs.)

Pyramid.—Let ABC be one of the six faces of a cube. Join each of these points to the middle point or centre of the cube, as in Fig. 157. It is evident that a square pyramid is formed, and also that the cube consists of six such pyramids, the base of each pyramid being one of the faces of the cube. Hence the volume of the pyramid $= \frac{1}{6}$ that of the cube.

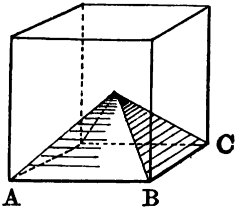


FIG. 157.

If h denote the height of the pyramid, then $h = \frac{a}{2}$ where a is length of side, BC of cube.

$$\therefore \text{ volume of pyramid} = \frac{1}{6}a^3, \text{ or } \frac{1}{3}a^2 \times \frac{a}{2} = \frac{1}{3}a^2h.$$

Hence *volume of pyramid* $= \frac{1}{3}$*(area of base × height)*,

or, the volume of a pyramid is one-third that of a prism on the same base and the same altitude.

Experimental Verification.—A model showing how to obtain readily the volume of a pyramid may be made as follows:

In Fig. 158, the four corners of the base $EFGH$ are joined to A, a pyramid is formed with A as apex and base that of the cube. It will be seen that three such pyramids can be formed from a cube, by means of three saw-cuts which are made to meet in one diagonal FA.

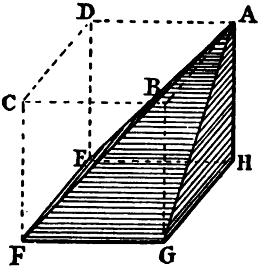


FIG. 158.

In this manner the cube is divided into three equal square pyramids. The height of each pyramid will be equal to the height of the cube, and the base of each, one of the faces of the cube.

Hence, *volume of each pyramid* $= \frac{1}{3}$ *(area of base × height)*.

The **surface area** of a pyramid is the area of the base which may be any polygon, together with the area of a number of triangles which form the faces or sides of the figure.

If the polygon forming the base be a regular polygon, each of the faces ABO, BCO, etc., of the solid (Fig. 159) consist of equal isosceles triangles.

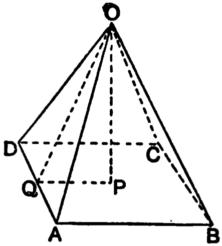


FIG. 159.—Surface of a pyramid.

The perpendiculars let fall from the vertex O to each side AB, BC, etc., will all be equal in length; the length may be found either graphically or by calculation. Let the length OQ be denoted by l, then:

$$\text{Area of each triangle} = \frac{a \times l}{2},$$

where a denotes the length of the equal sides AB, or BC, etc.

$$\therefore \text{ Lateral surface} = \left(\frac{a+a+\ldots}{2}\right) l = \frac{l}{2}\Sigma a.$$

And if n denote the number of sides in the base,

$$\text{lateral surface} = \frac{n \times a \times l}{2},$$

or *half the perimeter of the base multiplied by the slant height.*

∴ Total area = lateral surface together with the area of the base.

Ex. 1. Find the total surface area of a square pyramid. Side of base, 2 ft.; height, 3 ft.

The slant height $= \sqrt{3^2 - 1^2} = \sqrt{8}$;

$\therefore$ lateral surface $= \frac{8}{2} \times \sqrt{8} = 4\sqrt{8}$. Add area of base $= 4$ sq. ft.;

$\therefore$ total area $= 4\sqrt{8} + 4$ sq. ft. $= 15{\cdot}3$ sq. ft.

When the length of edge a and the height h are given the slant height $l = \sqrt{h^2 - \left(\frac{a}{2}\right)^2}$.

Frustum of a Pyramid.—When a pyramid is cut by a plane parallel to its base, the sloping faces are found to consist of a number of trapeziums.

As shown on p. 240 the area of a trapezium is the product of half the sum of the parallel sides and the distance between them.

Hence if in Fig. 160, l denote the sloping height of the frustum or trapezium $ABab$.

Area of trapezium $=\frac{1}{2}(ab+AB)l$.

If n denote the number of sides in the regular polygon forming the base,

$$\text{Lateral surface} = \frac{n}{2}(ab+AB)l,$$

$\therefore$ *Lateral surface = sum of perimeter of ends multiplied by half the distance between them.*

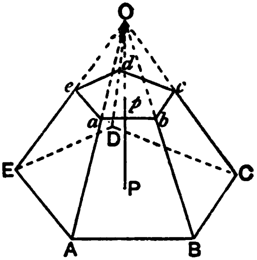


FIG. 160.—Frustum of a pyramid.

Cone.—As we have already seen, when the base of a pyramid is a circle, the solid is called a cone (Fig. 161), and is therefore a pyramid on a circular base, and as before:

Volume of pyramid
$= \frac{1}{3}$ *(area of base × altitude)*
$= \frac{1}{3}\pi r^2 \times h.$

Where r = radius of base,
and h = altitude.

Thus the volume of a cone is one-third that of a cylinder on the same base and the same altitude.

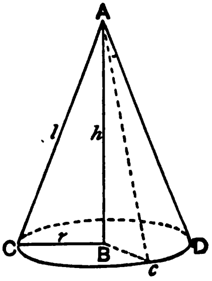


FIG. 161.—Cone.

Lateral surface of a Cone.—If the base of the cone be divided into a number of equal parts AB, BC, etc. (Fig. 162), then by joining A, B C, etc., to the vertex V, the solid is divided into a number of triangles, VAB VBC, etc.

If a line be drawn perpendicular to BC, and passing through V; and its length be p,

Then area of triangle $VBC = \frac{1}{2}(BC \times p)$.

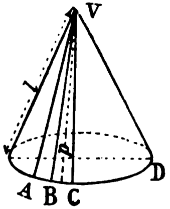


FIG. 162.—Lateral surface of a cone.

If n denote the number of triangles into which the base is divided, and a the length BC,

$$\text{Then. lateral surface} = \frac{n}{2} \times a \times p \text{ (approximately).}$$

As the number of parts into which the base is divided is increased, the product na becomes more nearly equal to the circumference of the base; and becomes equal to the circumference when the number of parts is indefinitely increased, also p becomes at the same time equal to l the slant height.

$$\therefore \text{ Lateral surface} = \tfrac{1}{2}\, 2\pi rl = \pi rl.$$

Or we may proceed as follows: Cut out a piece of thin paper to exactly cover the lateral surface of a cone. When opened out it will form a sector of a circle of radius l (Fig. 163).

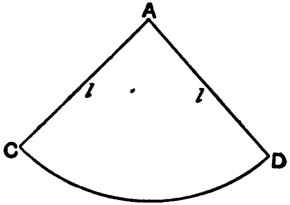


FIG. 163.—Development of a cone.

The length of arc CD
$=$circumference of base of cone.
$=2\pi r$.

But as we have seen on p. 247, the area of a sector is equal to half the arc multiplied by the radius,

$$\therefore \text{ Lateral surface} = \tfrac{1}{2}(CD \times l) = \tfrac{1}{2}(2\pi r \times l) = \pi rl.$$

Frustum of a Cone.—The area of the surface of a pyramid was found to consist of a number of trapeziums.

In a similar manner the curved surface of a cone may be assumed to consist of an indefinite number of trapeziums, the parallel sides forming parts of the circular ends, and the distance between them the length of the slant side.

$\therefore$ *Lateral surface* $= \tfrac{1}{2}$(sum of circumferences of ends × slant side).

$$= \tfrac{1}{2}(2\pi R + 2\pi r)l = \pi(R+r)l.$$

Where R is the radius of the base, and r the smaller radius.

To obtain the total area the areas of the two ends must be added, $\therefore$ Total area $=\pi(R+r)l+\pi(R^2+r^2)$.

EXERCISES. LXXI.

1. Find the volume of a pentagonal pyramid, side of base 7·25 ft., height 12·6 ft.

2. The slant side of a right cone is 10 ft., and its height 8 ft. Find its volume.

3. Find the volume of a right cone whose height is 210·6 ft., and radius of base $23\frac{1}{2}$ ft.

4. The slant side of a right cone is 20 ft., and its axis 16 ft. long. Determine its volume.

5. Find the volume of the frustum of a cone, the altitude being 9 ft., and the diameters of the ends 4 ft. and 2 ft. respectively.

6. Find the whole surface and solid content of a square pyramid, each side of the base being 12 ft., and the slant height 25 ft.

7. Find the content of a hexagonal pyramid; side of base 10 ft., perpendicular height 15 ft.

8. The axis of a hollow cone is 2·31 ft.; find the radius of its circular base, so that it shall contain 8·48 gallons.

9. The diameter of the base of a cone is 1 ft., and the distance of the vertex from the edge of the base is 8 in. Find the whole surface of the solid.

10. Find the number of cubic feet of air space in a conical tent 10 ft. high and 14 ft. diameter.

11. A regular pyramid with a hexagonal base has the perimeter of its base 30 ft., and its altitude 30 ft. Find its volume.

12. A cistern is in the form of a frustum of a pyramid, the internal dimensions are: length 10 ft., breadth 6 ft. at the bottom; and length 15 ft., and breadth 9 ft. at the top, if its depth is 4½ ft. find the number of gallons of water it will hold.

13. Find the volume of the frustum of a cone, the diameters of the ends 5 and 2, and height 4 ft.

14. Find the weight of a cast-iron hexagonal column 14 ft. high; 9 in. across the corners, and having a central longitudinal hole running through it 6 inches in diameter.

15. A reservoir, 100 ft. diameter at the top, 80 ft. diameter at the bottom, and 15 ft. deep, supplies a village of 150 inhabitants. The average consumption is 5 cub. ft. per person. How many day's supply will the reservoir hold?

16. The dimensions of a rectangular trough full of water are 10 ft. by 18 in., the depth is 10 in. How many times could a conical vessel, internal diameter 10 in., and 15 in. in depth, be filled from it?

Guldin's Theorems.—The preceding expressions for the areas and volumes of solids of revolution, and in addition others of a more difficult character, can easily be obtained by the two following theorems.

(1) *Area of Surface.*—The area of the surface traced out by the revolution of a curve about an axis in its own plane, *is equal to the product of the perimeter of the curve into the distance moved through by the centre of gravity of the curve.*

(2) *The volume* generated by the revolution of any plane figure about any external axis in its plane, *is equal to the product of the generating area and the distance moved through by the centre of gravity of the area.*

It will be seen that from (1) if the perimeter of the curve and the area traced out by it are known, we can find the position of the centre of gravity of the curve.

Also in a similar manner in (2) if the area of the figure and the volume traced out be by it known, the centre of gravity of the area can readily be obtained.

Surface and Volume of a Solid Ring.—A solid ring may be considered as a cylinder bent round until both ends meet, and the curved surface will obviously be equal to the circumference of its cross-section multiplied by the length (or mean circumference of the ring). Or, a circle with its centre C (Fig. 164), rotating about an axis AB, and connected to it by a thin bar CD, would trace out the surface and volume of a solid ring.

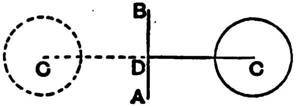


FIG. 164.

Let r denote the inner and R the outer radius of the ring, a the radius of the circle C.

The length (or circumference) of tracing curve $=2\pi a$.

Distance described in one turn $=2\pi\times DC$.

$$\therefore\quad \text{Area of surface}=2\pi a\times 2\pi\times DC=4\pi^2 a\times DC.$$

But $a=\dfrac{R-r}{2}$, also $DC=a+r$.

Hence, $\quad$ area of surface $=4\pi^2\left(\dfrac{R-r}{2}\right)\times\dfrac{R+r}{2}=\pi^2(R^2-r^2)$.

Volume of Ring.—The area of the tracing area $=\pi a^2$.

$$\therefore\quad \text{Volume}=\pi a^2\times 2\pi\times DC=\pi\left(\frac{R-r}{2}\right)^2\times 2\pi\left(\frac{R+r}{2}\right)$$

$$=\frac{\pi^2}{4}(R-r)^2\times(R+r).$$

The cross-section of the rim of a cast-iron fly-wheel is a square of 6 inches side. If the inner and outer diameters of the rim are 6 and 8 feet respectively, find (*a*) the area, (*b*) the volume, and (*c*) the weight of the rim.

(*a*) Perimeter of square $=4\times\frac{1}{2}=2$ feet;

$$\therefore\ \text{area}=2\times 2\pi\times 7=28\pi \text{ square feet.}$$

$$(b)\ \text{Volume}=(\tfrac{1}{2})^2\times 2\pi\times 7=\frac{7\pi}{2} \text{ cubic feet.}$$

$$(c)\ \text{Weight}=\frac{7\pi}{2}\times 450 \text{ lbs.}$$

In a similar manner the examples given on p. 273 may be verified.

Surface of a Sphere.—A semicircle of radius r, if made to rotate about a diameter as an axis, will trace out the surface of a sphere.

If G is the position of the centre of gravity and D the centre of the semicircle, then, as shown on p. 313, $DG=\frac{2r}{\pi}$.

Area traced out is the product of the length of the curve and the distance moved through by G.

Hence, $$\text{area}=\pi r\times 2\pi\times\frac{2r}{\pi}=4\pi r^2.$$

Volume of a Sphere.—Let AB be the base of a semicircle, radius r.

Its area $=\frac{\pi r^2}{2}$. The C.G. is at G, as shown on p. 313, where $DG=\frac{4r}{3\pi}$.

$$\text{Volume of solid}=\frac{\pi r^2}{2}\times 2\pi\times\frac{4r}{3\pi}=\frac{4}{3}\pi r^2.$$

EXERCISES. LXXII.

1. A hemispherical basin holds 1 gallon of water; find its diameter.

2. The base of a pyramid is a regular hexagon, each side $=40$ feet; what must be its height that its cubical content may be the same as that of a sphere whose radius is 21·5 feet?

3. If the diameter of a 9 lb. shot is 4 inches, find the diameter of a 12 lb. and a 24 lb. shot; and the thickness of a 13 inch shell which weighs 196 lbs.

4. A 36 inch iron shell is 4 inches thick; find its weight, given that a shot weighing 9 lbs. is 4 inches diameter.

5. A cylindrical hole 16 inches diameter is bored through the centre of a cube 32 inches edge; find the volume remaining.

6. Find the diameter of a spherical body which being thrown into a cylindrical tank whose diameter is 35 inches will make the water rise 2½ inches.

7. A cylindrical boiler, inside diameter, 5′ 6″, and inside length 29′ 0″. Find the contents in cubic feet and in gallons. Consider two cases, (*a*) when the ends are spherical, total length 34′ 6″, (*b*) ends flat, total length 29′ 0″. Also in the latter case find the quantity of water in gallons required to fill it to a height of 4 feet.

8. A spherical boiler, 10 feet diameter, is filled to a depth of 7 feet. Find the quantity of water in gallons.

9. A round tower, 120 feet high, is surmounted by a conical top 30 feet high. If the diameter of the tower be 50 feet find the whole exterior surface.

10. If the diameter of a dead-weight safety valve is 3 inches, the pressure of steam 120 lbs.; find the thickness of each of 12 cast-iron weights, 10 inches diameter, to give the necessary weight.

11. A triangular piece of plate, sides 3 feet 3 inches, 4 feet 9 inches, and 5 feet 2 inches respectively, weighs 2½ cwt. Find the weight of a square foot of similar plate.

12. Two buckets have the same depth of 8 inches. One is cylindrical 7½ inches diameter, and the other is a conical frustum, the diameter of its ends being 8½ inches and 6½ inches respectively. Which holds the greatest amount of water, and by how much?

13. A rectangular tank is 7 feet long by 1½ feet broad. Find how much the water will rise if a sphere of 9 inches radius be totally submerged in it.

14. A hemispherical dome, 50 feet diameter, is covered with sheet lead weighing 50 lbs. per square foot. Find weight of lead used.

15. A solid metal sphere, 5 inches in diameter, is formed into a tube 13 inches in external diameter and 3½ inches in length. Find the thickness of the tube.

16. A cylindrical bar of lead, the diameter of whose section is 2⅞ inches and whose length is 8½ inches, is melted down and cast into spherical bullets each ⅝ inch diameter. How many bullets can be so formed, and what will be the weight of each?

Summary.

The portion of space included under the three dimensions of length, breadth, and thickness is called the *cubical content*, the *volume*, or the *solidity* of a figure.

Total Surface of a Right Prism.—Perimeter of base multiplied by altitude, together with the areas of the two ends.

Volume of a Cube.—If a is the length of one side, the volume is a^3.

Volume of a Right Prism = *length* × *breadth* × *depth*
= *area of base* × *altitude*.

Volume of an Oblique Prism = *area of base* × *altitude*.

Volume of a Cylinder = area of base × height = $\pi r^2 \times h$.

Lateral Surface of a Cylinder = circumference of base × height
= $2\pi r \times h$.

Volume of an Oblique Cylinder = *area of base* × *altitude*.

Volume of a Hollow Cylinder = area of base × altitude = $\pi(R^2 - r^2)h$.

Volume of a Pyramid = $\frac{1}{3}$(area of base × height).

Convex Surface of a Right Pyramid = *half the perimeter of the base multiplied by the slant height.*

Volume of a Cone = $\frac{1}{3}$(area of base × height) = $\frac{1}{3}\pi r^2 \times h$.

Lateral Surface of a Cone = $\frac{1}{2}$(*circumference of base* × *slant height*).

Lateral Surface of Frustum of a Cone = $\frac{1}{2}$(sum of circumferences of ends × slant side) = $\pi(R + r)l$.

Surface of a Sphere of Radius $r = 4\pi r^2$.

Volume of a Sphere of Radius $r = \frac{4}{3}\pi r^3$.

Guldin's Theorems.—(1) The *area of the surface* traced out by the revolution of a curve about an axis in its own plane is equal to *the product of the perimeter of the curve and the distance moved through by the centre of gravity of the curve.*

(2) The *volume* generated by the revolution of any plane figure about any external axis in its plane, is equal to the *product of the generating area and the distance moved through by the centre of gravity of the area.*

CHAPTER XVIII.

LINEAR MOTION.

MOMENTUM. FORCE. MASS. WORK. ENERGY (POTENTIAL AND KINETIC).

Momentum.—When a given mass is in motion, we require to know not only the magnitude of the mass, but also its velocity. The product of the two, or quantity of motion of a body, is known as momentum, thus :

Momentum or quantity of motion of a body is the product of its mass and its velocity, or expressed as an equation.

Momentum = mass × velocity.

Unit of Momentum = (unit mass) × (unit velocity).

If the units are 1 pound and 1 ft. per second, then the unit of momentum is the momentum of a mass of one pound moving with a velocity of one foot per second.

A good example is furnished by the motion of a cannon and a shot. When a shot is fired from a cannon, the momentum generated in both will be the same, but the mass of the cannon is very much greater than that of the shot, hence the velocity or "recoil" of the cannon is much less than that of the shot, and is in the opposite direction.

Ex. 1. A cannon weighing 10 tons fires a shot weighing 600 lbs. The shot leaves the gun with a velocity of 1120 feet per second. If the cannon be free to move, find the velocity of recoil.

Momentum of cannon = momentum of shot ;

$$\therefore\quad 10 \times 2240 \times V = 600 \times 1120\,;\quad \therefore\quad V = \frac{600 \times 1120}{10 \times 2240} = 30 \text{ ft. per sec.}$$

Hence the cannon would recoil with a velocity of 30 ft. per sec.

Ex. 2. The shot in the preceding example, when its velocity is 1000 feet per second, passes through a plate weighing 5 tons, and goes on with a velocity of 200 feet per second. If the plate be free to move, find the velocity it acquires.

As the shot enters the plate with a velocity 1000 ft. per sec. and leaves it with a velocity of 200 ft. per sec.,

Loss of momentum in passing through plate $= 600(1000 - 200)$.

But this loss of momentum is acquired by the plate. Hence if V denote the velocity of the plate, we have:

$$V \times 5 \times 2240 = 600(1000 - 200);$$

$$\therefore \quad V = \frac{600 \times 800}{5 \times 2240} = 42\tfrac{6}{7}\text{ft. per sec.}$$

Force.—Suppose a body to possess a certain amount of momentum, it is impossible for it to alter its state of motion in any manner unless acted upon by something, and that something is termed *force*.

If a force F acting on a mass m during a time t changes the velocity from v_1 to v, then

$$Ft = m(v - v_1). \qquad \text{.........................(1)}$$

When the body is initially at rest, $v_1 = 0$;

$$\therefore \quad Ft = mv, \text{ or } F = \frac{mv}{t}; \qquad \text{.......................(2)}$$

or in words, force is rate of change of momentum, and if the time taken be one second we get:

Force is momentum per second.

Acceleration *when uniform is the change in the velocity per unit time.* In (1) the change in the velocity is $(v - v_1)$, denoting this by f, then

$$F = mf. \qquad \text{................................(3)}$$

This fundamental equation could also be obtained from (2) by writing for $\frac{v}{t}$ its equivalent f. As Eq. (3) is of the utmost importance it should be remembered in words thus:

Force = mass × acceleration,

or

$$\text{acceleration} = \frac{\text{force causing motion}}{\text{mass moved}}.$$

When the force F is variable its average or mean value is to be taken.

Uniform Force.—Under the action of a constant or uniform force the change in the momentum of a moving body will in any given second be the same as in any other second. Thus, the acceleration produced will be uniform acceleration. Such a uniform force is furnished by the attraction of the earth at any given place upon a mass at or near its surface. This force of attraction, as we have already found, is called the weight of a body.

The **weight** of a body is only another name for the **force** exerted by gravity on the mass of a body and varies at different places on the earth's surface. It also depends on the distance above sea-level (p. 15).

Although its numerical value at various places on the earth's surface is different, at any given place it remains practically constant, and produces a uniform acceleration of g units, hence :

If m denotes the mass of a body and W units of force represents the attraction of the earth. Then from Eq. (3) we have

$$W = mg, \text{ or } m = \frac{W}{g}.$$

or *the weight of a body is its mass multiplied by the number of units of acceleration produced by gravity in one second in a body moving freely towards the earth.*

A weight or force of 1 lb. produces g units of acceleration in unit mass (1 lb.). Hence unit force, called an *absolute unit*, is $\frac{1}{g}$ part of a pound—roughly, about half an ounce. This is called a *poundal*. A smaller unit of force is called a *dyne*, and we have the following definitions :

The Poundal is the name *given to that force, which, applied to a mass of one pound for one second generates a velocity of one foot per second.*

The Dyne is the name given to *the force, which, applied to a mass of one gram for one second generates a velocity of one centimetre per second.*

Gravitation or Engineers' Unit of Force.—What is called the gravitation unit of force is the force equal to 1 lb. Thus, consider the case of a string supporting unit mass (one pound or one gram) the tension in the string is directly equal and opposite to the downward force of gravity, hence the gravitation unit may be taken to be represented by the tension of a string supporting unit mass.

The choice of suitable units gives rise to so-called systems, thus:

The **F.P.S. System**, an abbreviation for foot, pound, second; or, as it is also called, the *British absolute system*. The unit of length is one foot, the unit of mass 1 lb., and the unit force is the *poundal*.

The **C.G.S. System** (or centimetre gram system) is very convenient on account of the decimal notation. Operations involving multiplication and division are readily performed. The objection raised by practical men to its use, is that the units are so very small that ordinary forces with which the practical man has to deal, require to be expressed by numbers inconveniently large.

Thus 1 dyne $=\frac{1}{g}$ gram $=\frac{1}{981}$ gram. $\therefore$ Weight of 1 gram $=981$ dynes.

Hence, to convert grams into dynes, multiply by 981. The dyne, as will be seen, is an exceedingly small force; the weight of 1 lb. $=4{\cdot}45\times10^5$ dynes (nearly).

EXERCISES. LXXIII.

1. A bullet weighing 1 oz. is fired from a gun weighing 10 lbs.; if the bullet leaves the gun with a velocity of 800 ft. per second find the recoil of the gun.

2. A projectile weighing 560 lbs. is fired from a gun weighing 40 tons with a velocity of 1600 ft. per second. Find velocity of recoil.

3. A bullet weighing an ounce is fired from a gun weighing 12 lbs. with a velocity of 800 ft. per second. Find the velocity of recoil.

4. A shot weighing $\frac{1}{2}$ ton is fired from a 100-ton gun with a velocity of 2000 ft. per second. Find the force necessary to stop the recoil in $2\frac{1}{2}$ ft.

5. If a piece of lead weighing 3 lbs. 12 oz. is moving at the rate of 500 yds. a minute, and a piece of oak weighing 2 lbs. 8 oz. at the rate of 30 ft. a second, what ratio does the momentum of the lead bear to that of the oak?

6. A hammer head of $2\frac{1}{2}$ lbs. moving with a velocity of 50′ per second is stopped in 0·001 second. What is the average force of the blow?

7. Investigate the equation of "impulse and momentum."

A mass of 1 lb. falls a distance of 3 ft. on to the head of a nail, which is sticking vertically in the horizontal face of a fixed piece of wood, and drives it in a third of an inch. Regarding the mass of the nail as negligible, find the average resistance of the wood and the time of penetration.

8. A ship of 2000 tons, moving at 3 knots, is stopped in one minute; what is the average retarding force? Neglect the motion of the water. One knot is 6080 ft., or one nautical mile, per hour.

9. A bullet mass 2 oz., moving at 800 ft. per second, strikes a target and is brought to rest in $\frac{1}{10}$ second. Find the average pressure exerted.

Work.—From Newton's first law of motion we know that a body at rest can only be set in motion by the action of force. Also, any change in the direction or speed of a moving body is produced by force. When a force acts in either of these ways it is said to do **work.** *The work done is measured by the product of the force into the distance through which it acts.* If F denote the force and S the space, the work done $= FS$.

Unless there is motion no work is done. Thus work is done by raising a mass from the ground, but no work is done by holding the mass stationary at any distance from the surface; in the latter case the mass could be placed on a suitable support, and although work must be done to place it on its support, the support does no work in the sense that no motion takes place.

Unit of Work.—In the measurement of work suitable units are requisite, the *unit of work* being *unit force multiplied by unit distance.*

The unit of work for many practical purposes is *the work done in raising a mass of one pound through one foot*; or, more generally, *the work done* in overcoming *a resistance of one pound through a distance of one foot,* and is called a **foot-pound** (ft.-lb.)

In the C.G.S. system the unit of work is the work done by a force of one dyne acting through a distance of one centimetre, and is called an **Erg.***

---

*Thus 1 lb. = 453·6 grams. 1 foot = 30·48 cm.

$$1 \text{ H.P.} = 550 \text{ ft.-lbs.} = \frac{550 \times 453{\cdot}6 \times 30{\cdot}48}{10^7} = 746 \text{ watts.}$$

$$1 \text{ ft.-lb.} = 1{\cdot}356 \times 10^7 \text{ ergs, nearly.}$$

It is obvious that the time taken does not enter into the estimation of work done. The same quantity of work is done whether an hour or a minute is spent in raising a weight to a given height. When the time taken in doing a certain amount of work, or in other words, when the *time-rate* of doing work is considered, we obtain what is called the **power** of an agent. The power would be measured by the quantity of work which an agent can perform in a given time, or **Power is the rate of doing work.**

Unit of Power.—The unit of power in general use by engineers is **horse-power,** meaning 33000 ft.-lbs. of work per minute, or 550 ft.-lbs. per second.

Watt.—In electrical work the unit of power, called a *watt*, is 10^7 *ergs* of work per second.

One horse-power (H.P.) is equivalent to 746 watts.*

* See footnote on previous page.

EXERCISES. LXXIV.

1. What work in foot-pounds is done in raising the materials for building a brick wall 50′ high, 12′ long, and 2′ 3″ in thickness, if one cubic foot of brick-work weighs 112 lbs.?

2. Define "force," "work," "foot-pound," and "horse-power."

A small metal planing machine, the table of which weighs 1 cwt., makes six backward and six forward strokes, each of $4\frac{1}{2}$ ft., in a minute, and the coefficient of friction between the sliding surfaces is 0·07. What is the work performed in foot-pounds per minute in moving the table?

3. A bucket when filled with water weighs 180 lbs., and is raised at a uniform rate from a depth of 150 ft. in eight minutes. Find the work done in one minute.

4. Explain fully the mode of measuring the work done by a force. What unit is adopted? A weight of 2 cwts. is drawn from a mine, 30 fathoms deep, by a chain weighing 1 lb. per linear foot. Find the number of units of work done.

5. In drawing a load a horse exerts a constant pull of 120 lbs.; how much work will be done in 15 minutes, supposing the horse to walk at the rate of 3 miles an hour?

6. The surface of the water in a well is at a depth of 20 ft., and when 500 gallons have been pumped out the surface is lowered to 26 ft. Find the number of units of work done in the operation, the weight of a gallon of water being 10 lbs.

The **Energy of a body** is its capacity for doing work, and consists of two fundamental forms, **kinetic** and **potential.**

The **Kinetic Energy** of a body is the energy which it possesses in virtue of its motion, and it is *measured by the work which has been done on the body to generate the motion*, or by the equivalent amount which the body can do against a force or resistance before its velocity is destroyed.

The acceleration f produced by a force F in a mass m is given by

$$f=\frac{F}{m}. \qquad \text{.................................(1)}$$

Also the velocity generated when a body moves from rest under the action of a constant force is given by $v^2=2fs$. ....(2)

If instead of f we write $\frac{F}{m}$ from (1), then

$$v^2=\frac{2Fs}{m}; \quad \text{or} \ \therefore \ Fs=\tfrac{1}{2}mv^2.$$

But Fs measures the work done by the force F acting through a distance s; hence the energy of a mass m in lbs., moving with velocity v, and which is measured by the work Fs done in producing it, is $\frac{1}{2}mv^2$.

The expression $\frac{1}{2}mv^2$ is very important, and is called the measure of the **kinetic energy** of the body.

The **Potential Energy** of a body is *the energy it possesses in virtue of its mass and its position.*

Thus a mass m requires an expenditure of mgh units of work to raise it to a height h, and so long as it remains in position, it is said to have *potential energy*, and may be made to do work as it falls towards the earth's surface.

Its potential then becomes kinetic energy, $\therefore mgh=\frac{1}{2}mv^2$.

There are many familiar forms of potential energy. By a few turns of a key we may obtain a bent spring which may be employed to drive the mechanism of a clock or watch, maintaining the motion for days or weeks.

A piece of metal or other substance stretched or compressed to any point within its *elastic limit* may be made to do work in recovering its original shape and dimensions.

The potential energy in compressed fluids may be utilized in 'lowing them to expand to their normal volume and pressure.

It should be noticed that Momentum *is*, but Kinetic Energy *is not* a vector; in each the mass m is necessarily positive and the positive or negative sign of mv will thus depend on that of v.

In kinetic energy, whether v is positive or negative, the square of v will be positive; hence the kinetic energy is independent of the direction.

Remembering that if m is the mass of a body, its weight, $W=mg$, we may replace the expression

$$mgh=\tfrac{1}{2}mv^2, \text{ by } Wh=\frac{Wv^2}{2g}.$$

The energy is then expressed in foot-pounds.

In certain cases the total work done will consist of two parts, one of which consists of resistance multiplied by space through which it acts, the other consists of kinetic energy generated.

Thus, if a horse moves a railway truck through a certain distance, the work done by the horse is the product of the mean force F exerted, and the distance s passed over; if the truck, when the force ceases to act, has a velocity of v feet per second, the work done on the truck is the mean resistance R multiplied by distance s, together with the kinetic energy generated;

$$\therefore\ Fs=R\times s+\frac{Wv^2}{2g}.$$

EXERCISES. LXXV.

1. What is meant by a time rate or a space rate of force?

2. Show that the work done by a force may be expressed as:

(*a*) Force × space.
(*b*) Momentum × average velocity.
(*c*) Impulse × average velocity.

3. A pendulum bob weighing 20 lbs. is suspended by a wire, the length from the point of suspension to the centre of the bob being 16 ft. The pendulum swings through an angle of 30° on each side of the vertical; find its potential energy when in the highest position, and its velocity when passing the lowest point.

4. The head of a steam hammer weighs 10 cwt., and has a fall of 8 ft. If it indent the iron on which it falls by 1 in. find the mean force exerted on the iron during compression.

5. Compare the force expended in pile driving by a ram or monkey of 1 ton falling 20 ft. with that of a weight of 2 tons falling 10 ft. If one blow of the former moves the pile 9 in., what is the average resistance that is opposed to its motion?

Angular Velocity.—When a point moves in any manner in a plane, the straight line joining it to any fixed point continually changes its direction; the rate at which such a straight line is rotating is called the *angular velocity* of the moving point about the fixed point. Angular velocity is *uniform* when the straight line connecting the moving and fixed points turns through equal angles in equal times, but variable when unequal angles are described in equal times.

Measurement of Angular Velocity.—Uniform *angular velocity* is measured by the number of radians through which the moving line turns in one second, variable *angular velocity* by the number of radians which would have been described in one second if the angular velocity at the instant referred to were maintained constant throughout the interval.

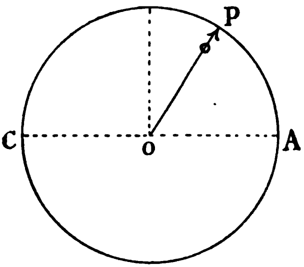


FIG. 165.—Angular velocity.

One of the most important cases of angular motion is when P (Fig. 165) is a point in a rigid body rotating about a fixed axis O. All points of the body move in circles, having their planes perpendicular to, and their centres in the axis. Hence, at any instant the angular velocity for all points of the body is the same.

If a point P is describing the circle APC of radius r with a uniform velocity of v feet per second, then, denoting the angular velocity by ω, the length of arc described in one second v, is the product of the angular velocity and the radius.

$$\therefore \quad v = r\omega, \text{ or } \omega = \frac{v}{r}.$$

In one revolution the moving point P describes a distance equal to the circumference of the circle. Hence, if t denote the time (in seconds) of a complete revolution

$$\omega = \frac{2\pi r}{rt} = \frac{2\pi}{t}.$$

In one revolution angle turned through is 2π, and in n revolutions $2\pi n$. From $v = \omega r$,

$$v^2 = \omega^2 r^2 = 2^2 \pi^2 n^2 r^2.$$

If the number of revolutions per minute is given, it is necessary to divide by 60.

Periodic Time.—If the point P (Fig. 165) is moving in the circle at uniform speed, the interval of time which elapses between two consecutive passages in the same direction through any point A is called the *periodic time.*

If T denote the periodic time, r the radius of the circle, as the circumference of the circle is $2\pi r$; $T = \frac{2\pi r}{v}$.

EXERCISES. LXXVI.

1. A wheel whose diameter is 5 ft. turns 40 times a minute; find its angular velocity, and the linear velocity of a point on its circumference.

2. The circumference of the earth being 25,000 miles, find the rate per second at which a place on the equator moves.

3. Define angular velocity. P is a point of a body turning uniformly round a fixed axis, and PN is a line drawn from P at right angles to the axis; if PN describes an angle of 375° in 3 seconds, what is the angular velocity of the body? and if PN is 6 ft. long what is the linear velocity of P?

4. What is the numerical value of the angular velocity of a body which turns uniformly round a fixed axis twenty-five times a minute?

5. A particle, whose mass is 3 lbs., moves uniformly in a circle; it describes the circumference 42 times a minute; find its angular velocity about the centre.

6. The radius of a fly-wheel in a stationary engine is $3\frac{1}{2}$ feet, and goes round uniformly 20 times a minute; find its angular velocity. At what rate per minute does the extremity move?

Curvilinear Motion.—The result of the action of force in linear motion when the direction of motion remains unchanged is, as already shown, to increase or diminish the magnitude of the velocity. But the action of force may continually alter the *direction* of a moving point, its *magnitude* remaining unchanged.

The path of the moving point will, under such circumstances, be in some curve, and the motion is said to be **curvilinear.**

The acceleration will be at right angles to the curve in which the body is moving; or in other words, the acceleration will be normal to the direction of the moving point.

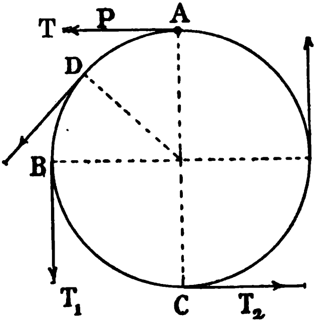


FIG. 166.—Uniform motion in a circle.

When a point P (Fig. 166) describes a circle at a uniform velocity, the direction of the moving point is continually changing. Thus at A it is in the direction of the tangent AT drawn to the circle at the point A; at B it is in like manner in the direction BT_1, and at C in the direction CT_2.

The acceleration, which in this case continually changes the direction of the moving point, is at right angles to the direction of motion of the moving point, and is therefore at any point A in the direction of the radius at the point; the magnitude remains unchanged, and is equal to the square of the velocity divided by the distance of the rotating point from the axis, or is $\frac{v^2}{r}$, where v denotes the uniform speed of the moving point, and r is the radius of the circle; or, denoting the acceleration by f, we have

$$f=\frac{v^2}{r}.$$

If a body of mass m describes a circle of radius r with uniform speed v, the *force* necessary to maintain the circular motion is obtained by multiplying the acceleration by the mass.

$$\therefore \quad mf=\frac{mv^2}{r}, \text{ or } F=\frac{mv^2}{r}.$$

The force in pounds is obtained by dividing by 'g';

$$\therefore \quad F \text{ lbs.}=\frac{mv^2}{gr}.$$

EXERCISES. LXXVII.

1. A body moves with a constant velocity in a given circle; state what is known as to the force which acts on the body.

Find a numerical result when the mass of the body is 10 lbs., and it moves at the rate of 900 ft. a minute in a circle 3 yards in diameter.

2. A body whose mass is 3 lbs. moves in a circle whose radius is 5 ft. with a velocity of 10 ft. a second; find the magnitude of the force which must act on the body, and state along what line and in what direction it must act.

3. A particle whose mass is 10 lbs. is constrained to move in a horizontal circle by a string 5 ft. long fastened to a fixed point; if at any instant the tension of the string is 98 pounds, find the velocity of the particle, and its angular velocity about the fixed point.

4. The centre of gravity of a body of 100 lbs. is revolving at 15″ from an axis at 250 revolutions per minute. What is its centrifugal force?

5. The mass of a particle is 10 lbs.; it is fastened to one end of a thread 6 ft. long, the other end of which is fastened to a fixed point; the particle is made to move in a vertical circle round the point in such a way that it passes through the lowest point of the circle with a velocity of 12 ft. a second; find the force (or forces) which at that instant are acting on the particle. Explain whether the "centrifugal force" is one of them.

Summary.

Momentum is the product of the mass and the velocity of a body $= mv$.

Unit of Momentum is the momentum of a mass of one pound moving with a velocity of 1 foot per second.

Force.—The change of momentum either produced, or which tends to be produced, is due to the action of force; *force is momentum per second.*

Unit Force is that force which acting for unit time produces either unit velocity or unit acceleration in unit mass.

The former is called a **poundal**, but the unit of force in general use by engineers is a force equal to the weight of one pound.

Force = mass × acceleration.

Work is the overcoming of resistance through a certain distance, and is measured by the product of the moving force and the distance through which it moves; ∴ work done = force × distance.

Unit of Work.—The usual unit is the **foot-pound**, that is, a force equal to the weight of one pound acting through a distance of one foot; other units are, in the C.G.S. system, a force of one dyne through one centimetre, called an **erg**.

Power is the rate at which work is done.

Unit of Power is 33,000 foot-pounds of work done per minute, and called 1 **horse-power**.

Energy is capacity for doing work, and may be kinetic or potential.

Kinetic Energy is one-half the product of the mass and the square of the velocity of a body.

Potential Energy, or energy of position, is the product of the mass of a body and the height to which it is raised.

Angular Velocity, when uniform, is measured by the number of radians through which any line (in a moving body) turns in one second.

If v is the linear and ω the angular velocity of a moving body of radians r, then $v=r\omega$.

The force in a radial direction necessary to maintain the circular motion of a mass m is given by $F=\frac{mv^2}{r}$, where v is the velocity in feet per second.

CHAPTER XIX.

THE SPECIFICATION OF VECTORS. POLYGON OF FORCES. FUNICULAR POLYGON. GRAPHIC CONDITIONS OF EQUILIBRIUM. MOMENTS OF FORCE. PARALLEL FORCES. BENDING MOMENT. CENTRE OF GRAVITY. ROOF TRUSSES. MOMENT OF INERTIA.

Vector Quantities.—Quantities which require for their complete specification enumeration of both magnitude and direction are called *vector* quantities, or shortly, *vectors.* Thus forces, velocities, accelerations, displacements, etc., are vectors, and may in each case be represented by a straight line.

To completely specify a vector we require to know

(1) Its *magnitude.*

(2) The *direction* in which it acts, or line of action.

(3) Its *point of application.*

The term *direction* applied to vector quantities is not sufficiently explicit. For example, in the specification of a vector the direction may be given as vertical, but a vertical direction may be either upward or downward, hence, what is called the **sense** of a vector is used. Thus if we include *sense,* four things require to be known before a vector is completely specified.

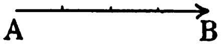


FIG. 167.—Representation of a vector.

The properties of a vector quantity may be represented by a straight line; thus, for example, a vector F acting at a point A can be fully represented by a straight line.

The length of the line to some convenient scale may represent the *magnitude* of the force. One end of the line A (Fig. 167) will represent the *point of application*, while the direction in which the line is drawn as from A to B will represent the direction or *sense* of the vector.

The *sense* is best represented by an arrow-head on the line.

Scalar quantities.—Those quantities which are known when their *magnitudes*, which are simply numbers, are given, such as masses, areas, volumes, etc., are called scalar quantities.

Addition and Subtraction of Vectors.—If two or more vectors of the same kind act in the same straight line and in the same direction, their resultant, or sum, is obtained by adding the numbers representing the magnitudes of the vectors together.

If the vectors act in the same straight line but in opposite directions, the resultant is equal to their difference, that is, to their algebraic sum.

When the vectors do not act in the same straight line the resultant is found by means of what is called the **parallelogram law**, which may be stated as follows: If two vectors not in the same straight line are represented in direction and magnitude by two adjacent sides of a parallelogram, the single vector called the sum or resultant which would produce the same effect as the two given vectors is represented in direction and magnitude by that diagonal of the parallelogram which passes through the same point as the two given vectors.

The sum of two vector quantities acting at a point may thus be obtained by drawing a parallelogram having the two vectors as sides. The diagonal of the parallelogram represents the sum, or resultant of the two vectors.

Ex. 1. Two vector quantities of 7 and 10 units respectively act at a point O at right angles to each other; find the sum or resultant and its inclination to one of them.

Set off a length $OA = 7$ units, and 10 units along OB, a line at right angles to OA. Complete the parallelogram, as shown in Fig. 168. The line joining O to C gives the resultant in direction and magnitude. This, by measurement, is found to be 12·1 units, and the angle $BOC = 35°$.

By calculation, $R^2 = 10^2 + 7^2$; $\therefore\ R = \sqrt{149} = 12\dot{\cdot}1$.

Tangent $BOR = \frac{7}{10} = \cdot 7$.

Using the slide-rule, or from Table VI., the angle is found to be 35°.

It will be at once seen on reference to Fig. 168 that the sum or resultant of the two vectors *OB* and *OA* could be obtained more easily by drawing a triangle *OBC*; when applied to find the resultant of two vectors the triangle *OBC* is called the **triangle of vectors.**

Thus *OB* is equal in magnitude and direction to one of the forces. *BC* is parallel to and similarly equal to the other, *OA*. The resultant is obtained by joining *O* to *C*. If a vector *CO* equal in magnitude but opposite in direction to *OC* be applied at *O*, the three vectors *OB*, *OA* and *CO* are in equilibrium.

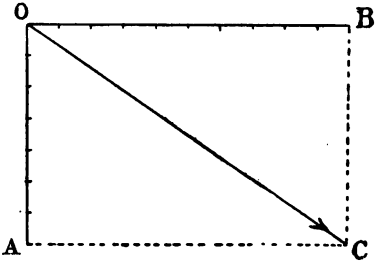


FIG. 168.—Parallelogram and triangle of vectors.

The vector *CO* is called the **equilibriant.**

Triangle of Vectors.—The three vectors acting at the point *O* (Fig. 168) are represented by the three sides of the triangle *OBC*, and it will be seen that the *magnitudes* and *directions* of the vectors are given by the sides of the triangle taken in order.

Hence, if three vectors acting at a point can be represented in direction and magnitude by the three sides of a triangle taken in order, the vectors are in equilibrium; or in other words, **the sum is zero.**

Conversely, if three vectors, acting on a particle be in equilibrium, they can be represented in direction and magnitude by the three sides of a triangle.

EXERCISES. LXXVIII.

1. Two forces of 8 and 12 units act at a point *A*, from *A* to *P* and *A* to *Q* respectively; *PAQ* is an angle of 72°; find by a construction drawn to scale the number of units in the force which balances them, and the number of degrees in the angles its direction makes with *AP* and *AQ*.

2. Draw a triangle ABC, whose sides BC, CA, AB are 7, 9, 11 units long. If ABC is a triangle for three forces in equilibrium at a point P, and if the force corresponding to the side BC is a force of 21 lbs., show in a diagram how the forces act, and find the magnitude of the other two forces.

3. Draw two lines AB and AC containing an angle of 120°, and suppose a force of 7 units to act from A to B and a force of 10 units from A to C. Find by construction the resultant of the forces, and the number of degrees in the angle its direction makes with AB.

4. ABC is a triangle whose sides BC, CA, AB are 7, 4, 5 units long; at A two forces act, one of 8 units from A to C, and one of 10 units from A to B; draw a straight line through A to represent their resultant in all respects, and state its magnitude.

5. State the rule for the composition of two velocities.

A ball moving at the rate of 10 ft. a second is struck in such a way that its velocity is increased to 12 ft. a second, and the direction of the new velocity makes an angle of 45° with that of the old velocity. Find by construction the velocity imparted by the blow and its direction.

6. Draw an equilateral triangle ABC, and produce BC to D, making CD equal to BC; suppose that BD is a rod (without weight) kept at rest by forces acting along the lines AB, AC, AD; given that the force acting at B is one of 10 units acting from A to B, find by construction (or otherwise) the other two forces, and specify them completely.

Mention the points that go to the specification of a force.

7. Find the resultant of two velocities of 9 and 7 ft. per second acting at a point at an angle of 120°.

8. Two accelerations of 5 and 4 ft. per second per second respectively act at a point at an angle of 45°. Find the magnitude and direction of the resultant acceleration.

One of the most useful applications of the triangle of forces is the case of an ordinary crane.

This machine usually consists of a *crane post* AB, a *tie rod* BC, and a *jib* CA (Fig. 169). When a weight is hung at a point C, the point C is kept in equilibrium by three forces. These are (*a*) the weight W, (*b*) the pull in the tie-rod, and (*c*) the compressive force, or the thrust in the jib. These forces may be represented on a convenient scale by the three sides of a triangle, taken in order. In Fig. 169 EF is parallel to, and equal in magnitude to, W; EG is parallel to the tie-rod, and FG parallel to the jib. The number of units of length in EG and FG, to the same scale on which EF is equal to W, will give the pull in the tie-rod and thrust in the jib respectively. The direction or sense of the forces in these bars is shown by placing arrow heads on *EF*, *FG*, etc. As the three forces are in equilibrium, the sense of the remaining two forces may be

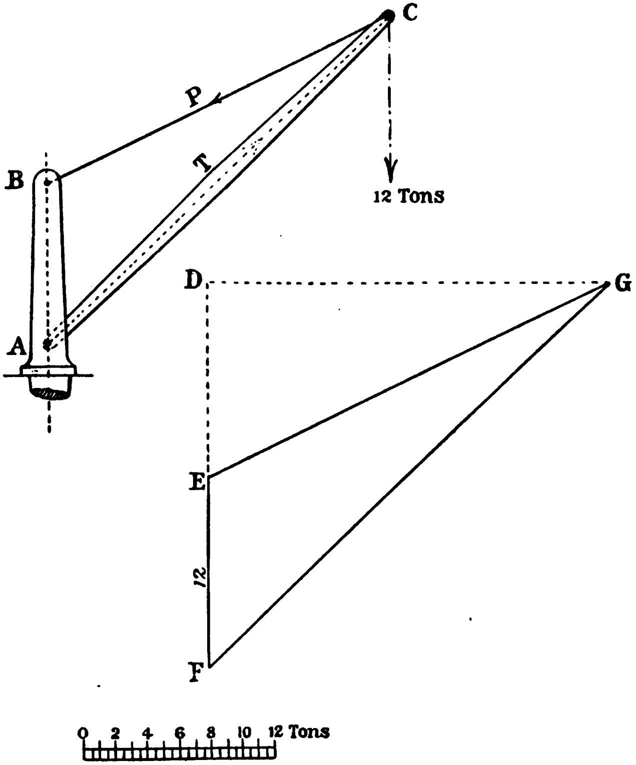


FIG. 169.—Ordinary crane.

furnished by following round the triangle in order, *E* to *F*, *F* to *G*, and *G* to *E*. The directions or sense of the forces in *AC* and *CB* can in this way be determined, and the force in the jib is recognized as one of compression, and that in the tie-rod as a pull.

Ex. 1. In a crane the dimensions are: Crane post 10 ft., length of tie rod *BC* 24 ft., and jib 31 ft. Find the pull in the tie-rod and the thrust in the jib, also the horizontal pull on the crane post when *W* is 12 tons.

In Fig. 169 a triangle *ABC* is drawn, using any convenient scale, such that $AB=10$ ft., $BC=24$ ft., and $AC=31$ ft. *EF* is made equal to 12 units on any scale, *EG* and *FG* are drawn parallel to *BC* and *AC* respectively. These lines intersect at a point *G*. The number of units in *EG* gives the pull in the tie-rod as equal to the weight of 28 tons; the number of units of length in *FG* the thrust in the jib equal to 35 tons. Now produce *FE* to meet a line drawn from *G* perpendicular to *FE* at *D*. Then *GD* gives the magnitude of the horizontal pull on the crane $=24\frac{1}{2}$ tons.

When, as in Fig. 169, the assumption is made that the load is hung from a point *C*, the triangle of forces can be applied to determine the pull in the tie-rod and the thrust in the jib, but the object of a crane is not merely to suspend a load from a

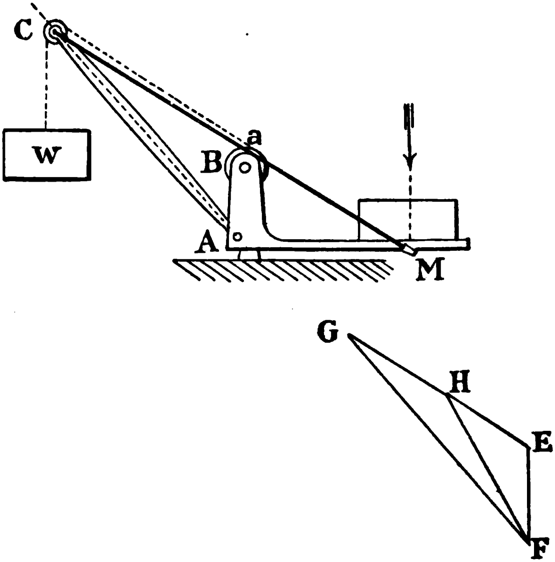


Fig. 170.—Wharf crane.

given point, but rather to raise or lower a weight. Hence a pulley is required at *C*, and a modification of the construction is required to take the tension of the chain into account.

In Fig. 170 a crane is shown in which the chain or rope supporting the weight is made to pass over a pulley at *C*, and is then led on to a drum or barrel, round which the rope or chain is coiled. Assuming the tension in the chain the same on each side of the pulley, *EF* is made equal in magnitude, and parallel in direction to *W*. *EH* is drawn parallel to the direction of the chain *BC* and equal to *W*. *HG* is made parallel to the tie-rod *BC*, and *FG* parallel to the jib and meeting *HG* at *G*. Then *HG* gives the pull in the tie-rod, and *FG* the thrust in the jib.

It should be noted that the chain may be led off from the pulley *C* parallel to the tie-rod, parallel to the jib, or in any intermediate position; this is effected by altering the position of the chain-barrel *a*.

It is very instructive and important to note the alteration produced in the pull in the tie-rod and the thrust in the jib when the direction of the chain is varied.

Instead of using the four forces *HE*, *EF*, *FG* of Fig. 170, it is sometimes convenient to find the resultant of the weight *W* and the tension of the chain, as shown by the line *FH*.

EXERCISES. LXXIX.

1. In a crane the pull in the tie rod inclined at an angle of 60° to the vertical is 18 tons. If the weight lifted be 8 tons, find the thrust in the jib.

2. The dimensions of a crane carrying a load of 6 tons are: crane post 6 ft., tie rod 12 ft., and jib 16 ft. Find by graphic construction the forces in all the bars.

(*a*) When the load is assumed as merely suspended from the point of intersection of tie rod and jib.

(*b*) When a pulley is used and the tension of the chain taken into account, the direction of the chain being (1) parallel to the jib, (2) parallel to tie rod.

(*c*) If two tie rods are used in the above, the distance between them at the place where they are attached to the crane post, equal to 12 in.

3. The post jib and tie rod of an ordinary wharf post crane form a triangle, the sides of which are 15, 45, and 50 ft. Find the forces acting in each member of the crane when a load of 5 tons is suspended over the pulley at the jib head, for the two cases when the chain passes from the pulleys to the barrel (*a*) parallel to the tie rod, (*b*) parallel to the jib.

4. The jib of a crane is subjected to a compressive force equal to the weight of 24 tons, the suspended load being 10 tons. If the inclination of the jib to the horizontal is 60°, find the tension in the tie rod.

5. The dimensions of a crane are, crane post 9 ft., jib 21 ft., tie-rod 16 ft., and the direction of the chain bisects the crane-post. The back-stay supporting the counterbalance is inclined at 45°. Find the forces in all the bars and the magnitude of the counterbalance acting at the foot of the stay, when 4 tons is lifted.

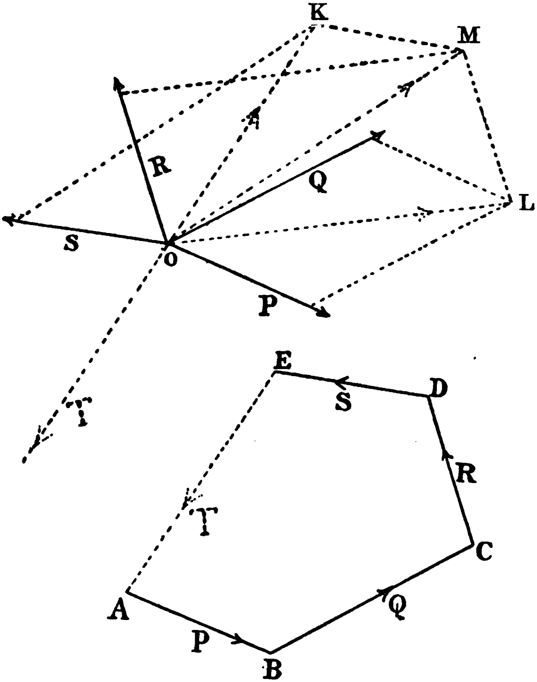


FIG. 171.—Sum of vectors acting at a point.

Sum of vectors acting at a point.—To find the resultant or sum of any number of vectors acting at a point.

Let $PQRS$ denote four vectors acting at the point O (Fig. 171) to find the sum or resultant.

The sum can be found by repeated applications of the parallelogram law. Thus, completing the parallelogram on P and Q as sides, the sum is given by OL. In a similar manner, combining OL and the force R, we obtain OM; this, combined with OS, gives a resultant OK, which is the resultant of the four given vectors.

Polygon of vectors.—A better and neater method than the preceding is to use the polygon of vectors.

From any convenient point A (Fig. 171), AB is drawn parallel in direction and containing as many units of length as there are units in the vector P. From the end B, BC is drawn parallel to Q, and equal in magnitude to it. Similarly, CD is made equal to R, and DE equal to S. The resultant in direction and magnitude of the four vectors is the line joining the initial point A to the final point E; and if a vector T, equal in magnitude but in the contrary direction, act at O, then the five vectors P, Q, R, S, T have a sum equal to zero, or the five vectors are in equilibrium. The polygon $ABCDEA$, when applied to forces, is called the **polygon of forces.**

Hence, if any number of vectors act at a point and can be represented by the sides of a closed polygon taken in order, the vectors are in equilibrium.

It is easy to prove by actually drawing the polygon to scale, that it is immaterial which of the five vectors we commence with. Also, if the direction of any one of the five vectors acting at O be reversed, it becomes the resultant of the other four.

Resolution of vectors.—By means of the parallelogram law we are able to replace two vectors acting at a point by a single vector which will produce the same effect.

It is often necessary to imagine a single vector replaced by two vectors acting in different directions. The two directions are usually assumed at right angles to each other.

Let OC (Fig. 172) represent in direction and magnitude a vector acting at a point O. If two lines OX and OY at right angles to each other be drawn passing through O, and BC and AC be drawn parallel to OY and OX respectively, we obtain two vectors OB and OA, which, acting simultaneously, produce the same effect on the point O as the single vector OC.

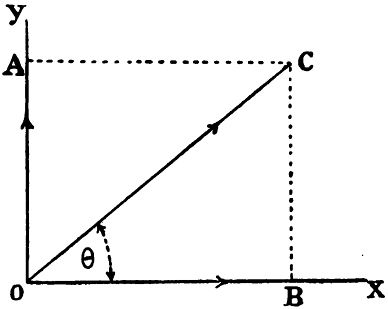


FIG. 172.—Rectangular components of a vector.

The two vectors OA and OB are called the *rectangular components* of OC, and the process of replacing a vector by its *components* is called *resolution.* The vector OC can be drawn to scale, and the components measured to the same scale. Or they can, by means of a slide-rule or logarithm tables, be easily calculated as follows:

Denoting the angle BOC by θ.

By definition (p. 160) $\frac{OB}{OC}=\cos\theta$; if θ and OC are known, then

$$OB=OC\cos\theta.$$

In a similar manner, $OA=OC\times\cos COA=OC\times\cos(90^\circ-\theta)$

$$=OC\sin\theta.$$

This important relation may be stated as follows: *The resolved part of a vector in any given direction is equal to the magnitude of the vector multiplied by the cosine of the angle made by the vector with the given direction.*

If the vector is a given force F, then the *resolved part of the force* in any given direction making an angle θ with the direction of the force is $F\cos\theta$.

EXERCISES. LXXX.

1. Draw AD, AE, lines containing an angle of 60°; draw AB bisecting the angle EAD; also draw AC at right angles to AB in such a way that AD falls within the right angle BAC. Let a force of 100 units act from A to D. By a construction drawn to scale, find the components of the force (a) along AB and AC, (b) along AE and AC.

2. Draw an equilateral triangle ABC, and draw AD at right angles to BC; forces of 5, 7, 9 units act from A to B, A to D, and C to A respectively. Find their resultant by the polygon of forces.

3. If four forces of 5, 6, 8, and 11 units make angles of 30°, 120°, 225°, and 300° respectively, with a fixed horizontal line, find the magnitude and direction of the resultant.

4. Forces P, $2P$, $3P$, and $4P$, act along the sides of a square taken in order. Find the magnitude, direction, and line of action of resultant.

5. Find the effect of a force of 46 lbs. in a direction making an angle of 32° with the line of the force.

[From a table of cosines, or by the slide rule, we have $\cos 32° = \cdot 848$. Hence, the effect (or component) $= \cdot 848 \times 46 = 39$ lbs.]

6. Define the rectangular components of a force.

Draw two straight lines OA, OB containing a right angle at O; within the right angle draw OP, such that AOP is an angle of 35°; a force of 18 units acts from O to P; find by construction its rectangular components along OA and OB.

7. A force acting on a body makes an angle of 60° with the horizon, and its horizontal component is known to be 1·35. Determine the force and its vertical component.

8. The horizontal and vertical components of a certain force are equal to the weights of 5 and 12 lbs. respectively. What is the magnitude of the force?

9. A force equal to the weight of 20 lbs., acting vertically upwards, is resolved into two forces, one of which is horizontal and equal to 10 lbs. What is the direction and magnitude of the other component?

10. A plane area of 70 sq. ft. is inclined at 35° to the horizontal. Find its horizontal and vertical projections.

Resultant of any number of forces in one plane, but not acting through one point.—In the case of forces acting at a point, the magnitude and direction of the resultant can be obtained from the force polygon, and since the resultant passes through the same point as the separate forces, its position is known.

When the forces do not act at the same point the magnitude and direction of the resultant are, as before, obtained from the force polygon; but the position, or line of action of the resultant has afterwards to be determined. This is effected by drawing, in addition to the *force-polygon*, what is known as the **funicular polygon.** The latter is very important in "Graphics," and we may with advantage consider its application to a few simple but important cases.

Given three forces P, Q, and R, which acting at different points on a rigid body do not meet at the same point when produced, to find the *resultant* and also its *point of application.*

Instead of denoting a force by a single letter, a very convenient and simple notation is to put a letter on each side of a force, thus, in Fig. 173, the force P may be denoted by the letters ab, Q by bc, and R by cd.

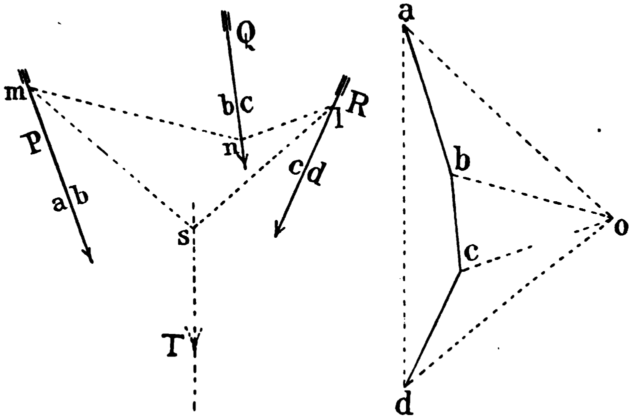


FIG. 173.—Forces not acting at a point.

In Fig. 173 ab, bc, and cd are drawn parallel to, and containing as many units of length as there are units of force in P, Q, R respectively; the resultant is given in direction and magnitude by the line joining a to d. But this *does not* determine its position. To find the *position* or *point of its application,* we choose any point o and draw radiating lines from o to a, b, c, d.

In the space b at any convenient point m draw a line mn intersecting the line of action of the force Q at n. In the space c draw a line nl parallel to oc intersecting the force R at l. Finally, draw lines ls, ms parallel to od and oa respectively, intersecting at s. This determines a point on the resultant, then through s, a line drawn parallel to ad, indicating a force numerically equal to it, is the resultant required.

The resultant reversed in direction is called the equilibriant, hence the four forces P, Q, R, and T reversed, would, if acting simultaneously, form a system of forces in equilibrium.

When the forces are in equilibrium the force polygon is a closed figure, and the graphic conditions of equilibrium become:

(1) *The force polygon must be a closed figure.*

(2) *The funicular or link polygon is a closed figure.*

Parallel Forces.—The funicular polygon construction applies to parallel forces. Let P and Q be two parallel forces acting at points A and B (Fig. 174). The force polygon abc is, of course, a closed figure; the resultant is $P+Q$, and is represented by ac in direction and magnitude.

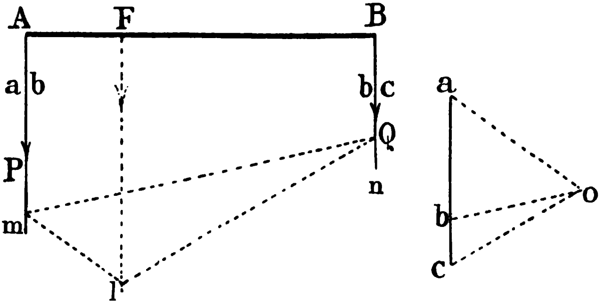


Fig. 174.—Parallel forces.

Join the points a, b, c to any convenient point O.

In the space b draw a line parallel to ob intersecting the forces P and Q at m and n respectively. From m draw a line ml parallel to oa, and a line nl parallel to oc intersecting at l, then l is a point in the required resultant. Draw lF intersecting AB at F, then F is the position of the resultant.

Centre of Gravity.—Every particle of a body is attracted by the earth, and the total force exerted constitutes the weight of the body. The separate forces due to the weights of the constituent particles are directed towards the centre of the earth, and are therefore practically parallel forces. The resultant of all these parallel forces, or the weight of the body will act through a definite point. This point is called the *centre of gravity* of the body.

The centre of gravity of symmetrical bodies is also their geometrical centres. Thus, the centre of gravity of a disc is the centre of its circular surface; of a rectangle, parallelogram, or square, the intersection of their diagonals.

To find the centre of gravity in the case of an irregular figure suspend it, as shown in Fig. 175. The centre of gravity is in the vertical line passing through the point of suspension. Now suspend it from any other point, again the centre of gravity is similarly in the vertical line passing through the point of suspension. Hence the centre of gravity of the face is at G the intersection of the two lines found. If the body is of uniform thickness the centre of gravity of the body is at the middle point of a line passing through G at right angles to its plane surface.

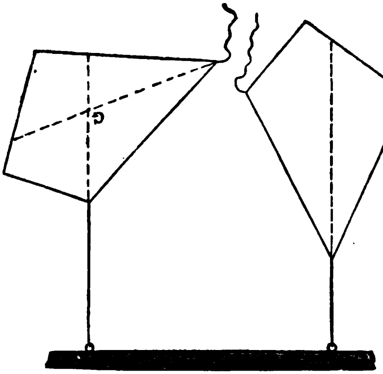


FIG. 175.—Centre of gravity of an irregular figure.

In a triangle any line drawn parallel to the base will have its centre of gravity at its middle point. Hence the centre of gravity of the triangle is in a line joining the vertex to the middle point of the opposite side.

In a similar manner, taking another line as base, the centre of gravity is in the line joining the apex to the middle point of the opposite side. The point of intersection of the lines so obtained is the centre of gravity of the triangle; it is easy to verify by measurement or by a simple application of Euclid, Bk. I., that the point of intersection is one-third the line measured from the base.

The centre of gravity of a curved line *ACB* (Fig. 176), which for experimental purposes may consist of a piece of round wire bent into the arc of a semicircle can be obtained in the following way. A piece of fine thread may be attached to the points *A* and *B*, and another piece from *C*, the middle point of the arc *ACB*, to *D* the middle point of *AB*. The centre of gravity is at some point in the line *DC*. To find its position, suspend the wire from any point *A*, and hang a plumb-line from the point of suspension, and notice the point where the plumb-line crosses the thread. This point is the centre of gravity, and *DG* will be found to be $\frac{2r}{\pi}$ where r is the radius *AD*.

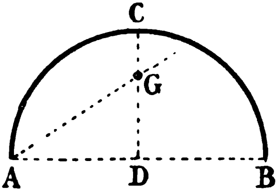


FIG. 176. –Centre of gravity of a curved line.

If instead of the wire used in the preceding experiment a semicircle of thin wood or cardboard be used, then the centre of gravity, or centre of area may be found thus: Draw the line *DC* as before; suspend the semicircle from any point *A*, and determine the point of intersection *G* of the plumb-line with *CD*, then *DG* will be found to be $\frac{4r}{3\pi}$, or the centre may be found by using a funicular polygon (Fig. 177).

If the two parallel forces *P* and *Q* (Fig. 174) be taken to represent the weights of two particles at a distance apart represented by *AB*, the point *F* is the centre of gravity of *P* and *Q*.

In a similar manner the centre of gravity of any number of masses can be ascertained by drawing the force and the funicular polygons. This assumes that one line such as *AB* passing through the centre of gravity is known. When the body is not symmetrical about a line, two force polygons and two funicular polygons are required. When applied to an area the *centre of area* is obtained.

Ex. 1. Find the centre of area of a cast-iron girder of the following dimensions: Flanges 3″ × 1″ and 9″ × 1″, depth of girder 12″.

To find the centre of gravity we divide the figure into three rectangles; those made by the two flanges *F* and *F′* and the web *W*.

The cross-section on a vertical line *ad*, Fig. 177, is drawn to any convenient scale, and lines *ab*, *bc*, *cd* are drawn at the middle, or centre of gravity, of each rectangle.

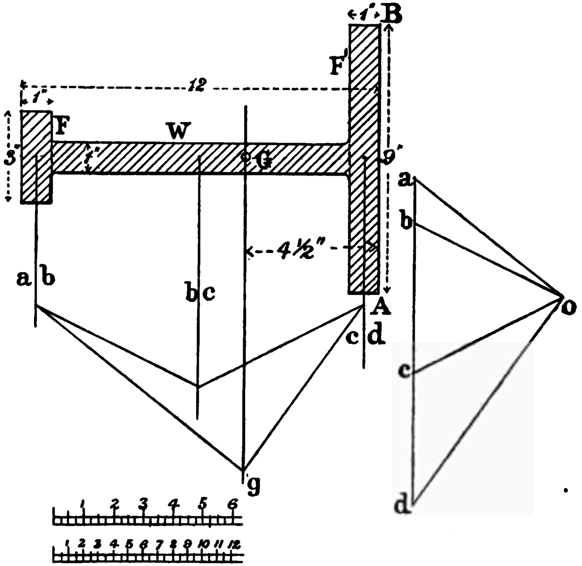


FIG. 177.—Centre of area.

The area of the rectangles *F*, *W*, and *F′* are 3, 10, and 9 square inches respectively. Hence in Fig. 177 make $ab=3$, $bc=10$ and $cd=9$ units. Join these to any convenient point (or pole) *O*. In the spaces *b* and *c* draw lines parallel to *Oa* and *Od* intersecting in *g*. Then a line through *g* meeting the axis at *G* gives the required centre of area of the section.

The mean area of a cross-section multiplied by the mean length is the volume of a body, this multiplied by the weight of unit volume gives the weight of the body; hence it follows that the *centre of area* just found is identical with the *centre of gravity*.

To obtain the centre of gravity by calculation we proceed thus:

Let $\bar{x}$ represent the distance of the centre of gravity from the base AB. Then $\bar{x}(9 \times 1 + 10 \times 1 + 3 \times 1) = 9 \times \cdot5 + 10 \times 6 + 3 \times 11\cdot5$;

$$\therefore \; 22\bar{x} = 99, \; \bar{x} = 4\cdot5 \text{ inches.}$$

Hence G is at a distance of 4·5 inches from the base AB.

The centre of gravity of an area in the shape of a semicircle can be determined by means of a funicular polygon.

Let ACB (Fig. 178) be the semicircle.

The centre of gravity is situated at some point in the line DC.

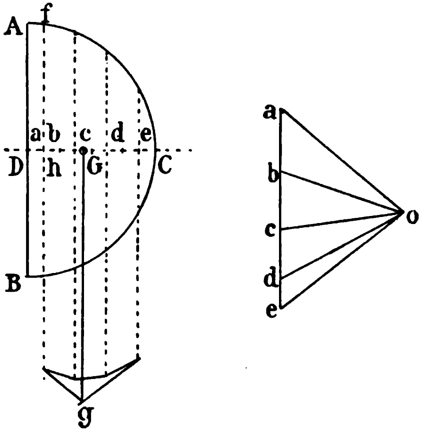


FIG. 178.—Centre of area of a semicircle.

Divide DC into any convenient number of equal parts or segments and draw lines perpendicular to DC and passing through the middle point of each, these lines are approximately the mean heights of the segments.

Make ab equal to fh' or some sub-multiple of it, and obtain in like manner the remaining distances bc, cd, de.

Join to any pole O, and as in previous examples draw the funicular polygon, intersecting at g; then a line through g, perpendicular to DC and meeting DC in G, is the centre of gravity required.

Moments of forces.—The *moment of a force* about a point is the product of the number of units in the force and the perpendicular distance from the point in units of length.

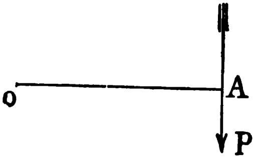


FIG. 179.—Moment of a force.

Thus let P be the given force and O the given point (Fig. 179), on the direction of P let fall from O a perpendicular OA; then the *moment of P about* $O = P \times OA$.

The effect of the force depends not only upon the magnitude of the force but upon the distance of the point or the distance OA.

Ex. 1. Let P (Fig. 179) be a force of 10 lbs. weight, and $OA = 24$ inches; then

moment of $P = 10 \times 2 = 10$ ft.-lbs., or 240 inch-lbs.

In a similar manner, if several known forces were acting at given distances from a point O, then the moment of each force would be its magnitude multiplied by the perpendicular distance from the point. When in this manner the moment of each force has been obtained the algebraic *sum* could, taking the direction of each force into account, be obtained by addition.

The moments tending to turn the body in the same direction as the hands of a watch would be subtracted from those tending to turn the body in the opposite direction. The usual convention is that the former moments are negative, the latter positive.

Bending Moment.—*The bending moment at any given section is the algebraical sum of the moments of all the forces acting on one side the section.* A load W is supported at the centre of a beam or girder resting on supports at the ends of the beam (Fig. 180). To determine graphically the bending moment at any section we may proceed as follows:

Using two scales, a *linear scale* for the given length, and a *force scale* on which bc is made equal in magnitude to W. A point, or pole o, is selected such that oh, a horizontal line drawn from o and meeting bc in h, is 5 or 10, or other convenient number *on the linear scale.* As will be seen later, 10 or some multiple of it is to be preferred.

Let R and R' denote the pressures, or reactions, on the two supports.

It is convenient to letter the forces as in Fig. 180, and proceed to set off the given force bc; next join b and c to o.

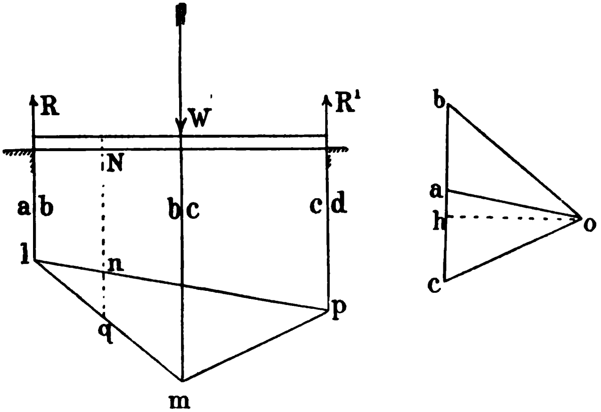


Fig. 180.—Bending moment.

Selecting any point m in the line of action of W, draw ml parallel to ob and mp parallel to oc; join l to p, then a line oa drawn from o parallel to lp will determine the magnitudes of the two reactions ca and ab. It will be found that the point a, the intersection of oa with bc, is at the centre of bc indicating (what we could infer otherwise) that the two reactions R and R' are equal in amount.

The bending moment at any section N is by definition equal to $R \times AN$.

Now from the similar triangles lnq, aob,

$$\frac{nq}{ab} = \frac{ln}{oa} = \frac{AN}{oh};$$

$$\therefore \quad nq \times oh = R \times AN.$$

Or the bending moment at N is the product of the ordinate nq multiplied by the length of oh. If oh is made equal to 10 or some multiple of it, then in multiplying nq by oh it is only necessary to alter the decimal place.

Ex. 1. A beam is supported on two walls 12 ft. apart. A weight of 120 lbs. is hung at C, a distance of 3 ft. from one wall, and a weight of 240 lbs. at D, 3 ft. from the other wall. Neglecting the weight of the beam, find the bending moments at D and C, also at M, the middle of the beam.

As in the last example, we obtain the reactions 210 lbs., and 150 lbs.

The bending moments at D, M, and C are 630 ft. lbs., 540 ft. lbs., and 450 ft. lbs. respectively.

In Fig. 181 a simple couple-close roof truss is shown carrying a load of 280 lbs. at the ridge. Determine the magnitude and nature of the forces in the bars 1, 2, 3, also the two reactions R and R'.

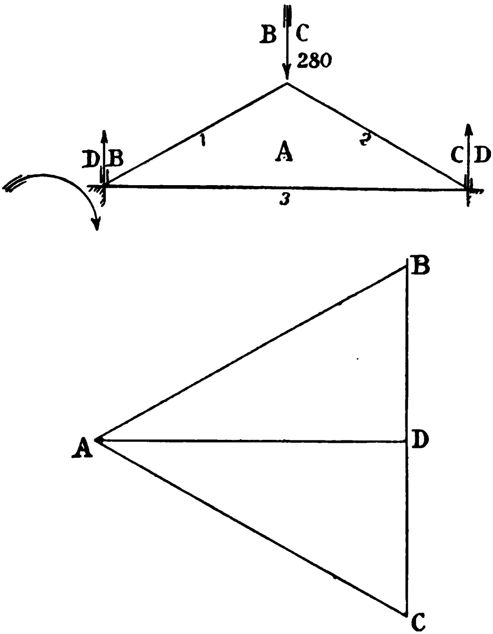


FIG. 181.—Couple-close roof truss.

The frame diagram is first drawn to any convenient scale. In the force polygon *BC* is made to scale equal to 280 lbs.

The two reactions, when unequal in amount, may be determined by drawing a funicular polygon. In the present case they are obviously equal and *CB* is consequently bisected at *D*, the two reactions being represented by *CD* and *DB*.

To find the magnitude and nature of the forces in the bars, it is only necessary to consider that three forces are acting at each joint of the truss where two bars meet, and as one force is known in magnitude and direction, and the other two in direction, the triangle of forces for that joint can be drawn. In the above case it is immaterial which joint of the three is considered first. This, however, is not often the case; in a more complicated truss, four or more forces may act at some of the joints, if, as is very common, the magnitude of one is known, and only the directions of the others, the polygon of forces for the joint cannot be constructed. It will be found convenient to commence in all cases by finding the reactions and the forces acting at a joint where there are not more than two unknowns to be determined.

A letter *A* placed inside the frame diagram enables the diagrams for all the bars to be drawn. Thus, considering the forces acting at the left-hand support, and reading off the known force *DB* upward, we read off in watch-hand direction the forces *BA* and *AD*.

To obtain their magnitude and nature, draw *BA* parallel to bar 1 through *B*, and *AD* parallel to bar 3 through *D* and intersecting it at *A*. The triangle *ABD* is a triangle of forces for the joint. The number of units of length in *BA* and *AD* on the same scale on which *BD* is equal to *R* gives the magnitudes of the forces in bars 1 and 2 respectively.

To ascertain the nature of the forces, the forces are read off as indicated by the arrow in the upper part of Fig. 181, *i.e.*, in watch-hand direction. Thus, the force *BA* is directed towards the joint, and *AD* away from it; hence in the bar 1 the force is compressive, and in bar 2 a *tensile* force. At the next joint we read off *BC* downwards. A line joining *C* and *A* should be parallel to bar 2. This fact concerning the closing line of the force polygon gives a check on the accuracy of the work. The forces may be read off as before in watch-hand direction—*B* to *C* downwards, *C* to *A* and *A* to *B*, indicating that the force in bar 2 is compressive and that bar 1 is in compression, conclusions which confirm the previous result.

In a similar manner by reading off the forces at the remaining joint, all the lines belonging to which are already drawn, the previous results are again confirmed.

EXERCISES. LXXXI.

1. In Fig. 182 a roof truss is shown having a load of 2 tons on each joint. Find the reactions, also the forces in all the members.

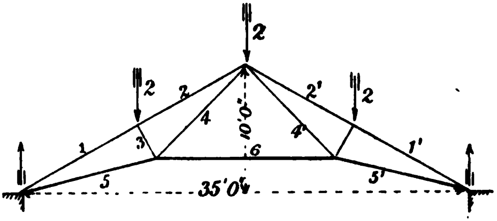


FIG. 182.

2. In a Warren girder as in Fig. 183, find the forces in all the members and their directions.

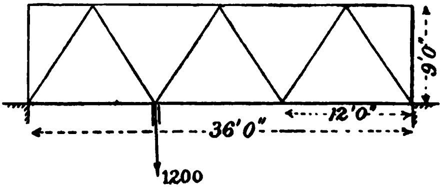


FIG. 183.

3. A king post roof truss of 28 ft. span and 7 ft. rise is loaded with 1200 lbs. on each of three joints, and in addition with 1000 lbs. at foot of the king post Fig. 184. Find the forces in the members of the truss.

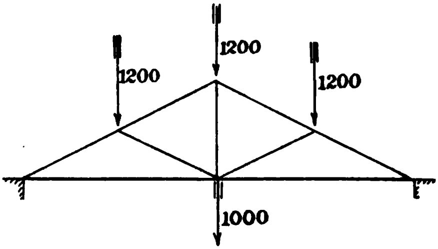


FIG. 184.

Moment of Inertia.—The moment of inertia of a series of particles m_1, m_2, etc., at distances x_1, x_2, etc., from an axis is the quantity

$$m_1x_1^2+m_2x_2^2+\dots \text{ or } \Sigma mx^2.$$

The term **axis** is usually applied to the line about which the moment of inertia is taken.

When dealing with a rigid body instead of a system of particles, we may imagine the body divided into a very large number of parts, each part so small that it may be regarded as a single particle.

Hence *moment of inertia of body* $=\Sigma(mx^2)$.

The summation symbol Σ is used to denote the sum of all the terms $m_1x_1^2$, $m_2x_2^2$, etc.

It should be noticed that it is not necessary to consider the signs of the various distances, since all negative distances on being squared give a positive result.

By simple applications of the integral calculus the moment of inertia (usually indicated by the symbol I) of some of the sections frequently used can be obtained.

The *moment of inertia of a rectangle* occurs so frequently that the student will find it advisable to remember the following facts:

Let b denote the breadth and d the depth of a rectangle; its moment of inertia about an axis o, o, passing through its centre of area may be denoted by I_o. Then $I_o=\frac{1}{12}bd^3$.

Let I_A denote the moment of inertia about an axis AA, parallel to oo, and at a distance x from it, then I_A is equal to *the moment of inertia about the former axis together with the area of the section multiplied by the square of the distance between the axes;*

$$\therefore\quad I_A=I_o+bd\times x^2=\tfrac{1}{12}bd^3+bdx^2.$$

Ex. 1. Find the moment of inertia of a rectangle, breadth 3 inches, depth 2 inches,

(*a*) about an axis passing through its centre of gravity;

(*b*) about an axis parallel to the former, but at a distance 10 inches from it;

(*a*) $I_o=\frac{1}{12}\times3\times2^3=2$ units.

(*b*) $I_A=2+3\times2\times10^2=602$ units.

Ex. 2. Show that the moment of inertia of a rectangle, sides b and d, about one edge is $\frac{1}{3}bd^3$.

Here x, the distance between the axes, is $\frac{d}{2}$;

$$\therefore \quad I_A = \frac{1}{12}bd^3 + bd \times \left(\frac{d}{2}\right)^2 = \frac{1}{3}bd^3.$$

The majority of sections of girders and other important structures may be readily divided into rectangles, and the moment of inertia found either by calculation, or by graphic methods.

To obtain the moment of inertia of a given section, the moment of inertia of each rectangle into which the figure may be divided is obtained, and the sum of all these separate moments when added together will give the total moment of inertia of the section. The most convenient axis about which to find the total moment of inertia, is the line passing through the neutral axis, or centre of gravity of the section.

When the moment of inertia is obtained graphically two funicular polygons require to be drawn, care being necessary to ensure accuracy. The method adopted can be best shown by an example.

Ex. 1. Find the centre of gravity and moment of inertia of the section of a cast-iron girder, the dimensions are: flanges, 3″ × 1″ and 9″ × 1″; web, 10″ × 1″.

Centre of Gravity.—Having drawn to a convenient scale the section of the girder, we next divide the web into five equal parts as shown, and at the centre of each interval, and also at the centre of each flange, lines are drawn as shown in Fig. 185.

Set off *ab*, *bc*, etc., equal on the force scale to the number of units in the section or segment to which they refer; join a, b, c, etc., to point o, make $ho = 5$ units, and draw a funicular polygon having its sides *em*, *mn*, *no*, *op*, *pq*, and *qr* parallel to *bo*, *co*, *do*, *eo*, etc. The final sides re', $e'e$ drawn parallel to *ok* and *oa* respectively determine G the centre of gravity of the section.

Moment of Inertia.—Produce all the sides of the funicular polygon e, m, n, o, p to intersect the line Gy passing through G at e', m', n', o', and p', and join these points to a second pole o', making the distance $oh' = oh$.

Draw a second funicular polygon e'', m'', etc., the initial and final points of which are e'' and r''; draw lines from e'' and r'' intersecting the vertical through G at y and X; measure on the force scale the distance Xy.

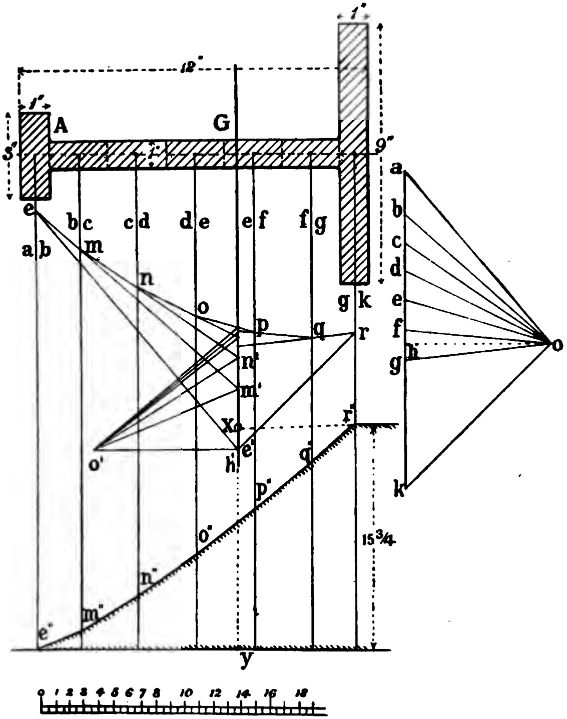


FIG. 185.—Moment of inertia obtained graphically.

Then moment of inertia, $I_o = Xy \times oh \times oh'$.

In the above example Xy is found to be 15·75, hence

$$I = 15{\cdot}75 \times 5 \times 5$$
$$= 393{\cdot}75 \text{ units.}$$

Ex. 2. Find the moment of inertia of a rolled steel girder; flanges, 4″ × ½″; depth of beam, 10″.

As the section is symmetrical the centre of gravity is at the middle of the web.

Moment of Inertia.—This is obtained as in the last example; the same notation may be used, lines drawn from the initial and final points of the second funicular polygon, perpendicular to Gy, meet Gy in points X and y.

The distance Xy measured on the force scale will be found to be 4·75; $\therefore\ I = 4{\cdot}75 \times 5 \times 5 = 118{\cdot}75$ units.

EXERCISES. LXXXII.

1. Section of a cast-iron girder; span 22 ft. (Fig. 186). Find (*a*) its weight; (*b*) position of centre of gravity; (*c*) its moment of inertia.

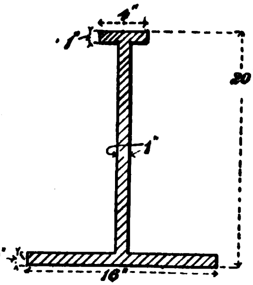


FIG. 186.

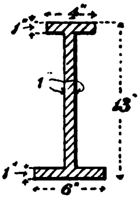


FIG. 187.

2. Cast-iron girder for 17 ft. span (Fig. 187). Find its weight, position of centre of gravity, and moment of inertia.

3. Flanges of a cast-iron girder are 4″ × 1″ and 6″ × 1″, depth, 10″; 15 ft. span. Find its weight, position of centre of gravity, and moment of inertia.

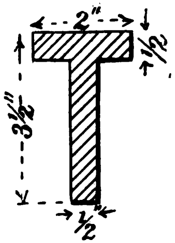


FIG. 188.

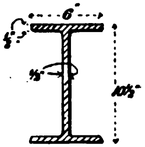


FIG. 189.

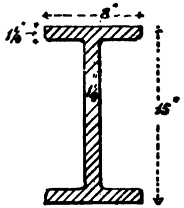


FIG. 190.

4. A rolled angle iron section is shown in Fig. 188. Find its weight per foot length, position of centre of gravity, and moment of inertia.

5. A section of a rolled steel joist is shown in Fig. 189 for 12 ft. span. Find its weight and moment of inertia.

6. Rolled iron girder; length 16 ft. **Find its weight and moment of inertia (Fig. 190).**

7. Rolled iron girder; length 15 feet, flanges 8″ × 1½″, depth 24″. **Find its weight and moment of inertia.**

8. Wrought-iron joist; flanges 6″ × 1″, depth 12″, length 18 ft. **Find its weight and moment of inertia.**

Summary.

Vector Quantities are quantities requiring for their complete specification both *magnitude* and *direction*. *Scalar quantities* only involve magnitude.

A Vector is specified when its *magnitude*, *direction*, *point of application*, and *sense* are given.

Addition of Vectors.—The resultant or sum of two or more vectors acting in the same straight line is obtained by adding together with their proper signs the numbers representing the magnitudes of the vectors.

Parallelogram Law.—When two given vectors do not act in the same straight line they are compounded by drawing a parallelogram having its adjacent sides equal and parallel to the two forces, the diagonal of the parallelogram then represents the sum of the two vectors.

Triangle of Vectors.—If three vectors acting at a point can be represented in direction and magnitude by the three sides of a triangle taken in order, the vectors are in equilibrium, or, their sum is zero: the converse is also true.

Sum of any Number of Vectors at a Point.—The sum of any number of vectors acting at a point can be found by repeated applications of the parallelogram law, or better by the **polygon of vectors.**

Resolution of Vectors.—Any single vector can be replaced by two vectors acting at an angle, the angle usually taken is a right angle, and the two vectors are called **rectangular components.**

The Component (or effect) **of a Vector in a given direction** is the product of the magnitude of the vector and the cosine of the angle made by the vector with the given direction.

Forces not acting through one point.—The magnitude and direction of the resultant is obtained from a **force polygon** and its position by a **funicular polygon.**

The Centre of Gravity of a rigid body is the point upon which the body could be balanced; when a body is suspended from a point and is free to move, the centre of gravity is in the vertical line passing through the centre of suspension.

Moment of a Force about a point is the product of the force, and the length of the perpendicular let fall from the point, on the direction of the force.

Bending Moment at a Section is the algebraical sum of the moments of all the forces on one side the section.

Moment of Inertia of a body $=\Sigma mr^2$ where the summation sign includes the masses of all the parts into which the body may be divided, multiplied in each case by the square of the distance of each component mass from the centre.

CHAPTER XX.

USE OF SQUARED PAPER.

PLOTTING. INCH—CENTIMETRES. SIMPLE MACHINES. AREA OF DIAGRAM TO REPRESENT WORK DONE BY A FORCE. WORK DONE BY A VARIABLE FORCE. COMPRESSION OF SPRINGS. STRESS AND STRAIN. MODULUS OF ELASTICITY. INDICATOR DIAGRAMS.

Use of Squared Paper.—Two quantities, the results of a number of observations or experiments, which are so related that any alteration in one produces a corresponding change in the other, can be best represented by a **graphic method**, in which it is possible to ascertain by inspection the relation that one variable quantity bears to another.

For this purpose we obtain *squared paper* or paper having equidistant vertical and horizontal lines $\frac{1}{10}''$, $\frac{1}{4}''$, 1 mm., etc., apart; these cover the surface of the paper with little squares (Fig. 191).

Commencing at the lowest left-hand corner of the paper, the lowest horizontal line may be taken as one axis or line of reference, and the vertical line nearest the left edge of the paper as the other axis.

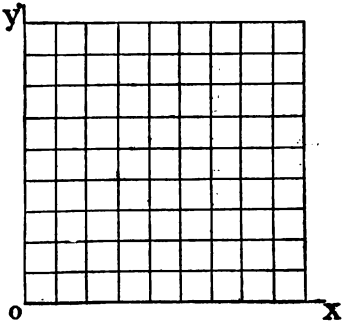


FIG. 191.—Squared paper.

The two lines *ox* and *oy* at right angles are called **axes.** The horizontal *ox* is called the axis of **abscissae**; *oy* the axis of **ordinates.**

To plot a series of observations we proceed as follows: Along *ox* one or more squares are used to represent a unit of one of the two quantities, and distances along the vertical line *oy* to represent the other.

In this manner a series of points may be obtained; these are usually marked with a small cross or circle and may be joined by a light line. The line so drawn connecting the points may form a fairly uniform curve, but if, as is often the case, the line is irregular and broken, a uniform curve lying evenly among the points should be drawn (this may be done by bending a thin piece of wood). The curve so drawn represents the average value of the results and serves to check errors of observation. In the case of experimental results, the points are arranged where possible to lie as nearly as possible on a straight line. When the results are plotted as described, the line which most nearly agrees with the results may be obtained by using a piece of black thread, this stretched and placed on the paper can be moved about as required until a good average position lying most evenly among the points is obtained. A better plan is to use a **strip of celluloid** on which a line has been drawn, the transparency of the strip allowing the points underneath it to be clearly seen. When the position of the line is determined by its extremities the line is inserted by using a straight edge or the edge of a set square.

It should be carefully noted that the word *curve* is used to denote *any line, whether straight or curved,* representing the relation between two quantities.

The advantage due to this method is obvious; although from a series of numbers an average value may be obtained by the ordinary arithmetical methods, any gradual change, and more especially the relation of the change to the result, is either disregarded or may not be noticed.

The following examples will serve to illustrate the methods used:

Ex. 1. *To ascertain without calculation the number of centimetres in a given number of inches; or conversely, the number of inches in a given number of centimetres.*

From p. 5 we may obtain the following list:

Inches.	Centimetres.
1	2·54
2	5·08
2½	6·4
3½	8·9

Using the vertical axis *oy* to denote inches and the horizontal axis *ox* to denote centimetres (Fig. 191), we read off 2·54 on the horizontal axis and the first division on the vertical, so obtaining a point on the curve required. Another point is the intersection of two lines drawn from 5·08 horizontal and 2 vertical.

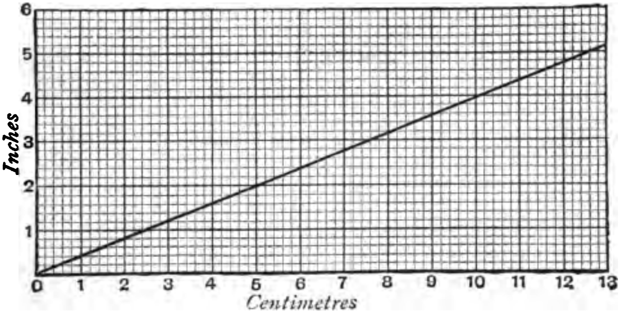


FIG. 192.—To ascertain by inspection the number of centimetres in a given number of inches.

In a similar manner other points may be obtained; a fine line drawn through the points enables any intermediate value to be found. The result is shown in Fig. 192; hence, the equivalent value in centimetres of any given number of inches, or conversely, the equivalent in inches of any given number of centimetres, can be found by inspection by means of a curve or line of this kind.

In a similar manner, if the divisions on the vertical or horizontal axis are made to denote square inches and those along the other axis square millimetres, the conversion from one to the other is readily made. The relation between pounds and kilograms can be obtained in a similar manner.

Equation of a Line.—In the preceding example, the position of a point is ascertained if we measure along the horizontal axis a distance equal to 2·54 units, and a distance of one unit along the vertical axis. The vertical distance is called the y co-ordinate, and the horizontal distance the x co-ordinate of the point. In a similar manner the co-ordinates of another point are $x=5{\cdot}08$, $y=2$. When, as in the preceding example, the relation between the two variables can be represented by a straight line we can obtain what is called the *equation of the line.*

The equation of the line is of the form

$$y=ax+b, \qquad \text{.................................(1)}$$

where a and b are constants, then, if in (1) simultaneous values of x and y are inserted, the values of a and b can be found.

Thus when $y=1$, $x=2{\cdot}54$;
also when $y=2$, $x=5{\cdot}08$.

Substituting these values in (1) we obtain

$$\begin{array}{l} 2=a\times 5{\cdot}08+b \\ \underline{1=a\times 2{\cdot}54+b} \end{array}$$

by subtraction

$$1=a\times 2{\cdot}54$$

$$\therefore \quad a=\frac{1}{2{\cdot}54}={\cdot}39.$$

Also substituting this value for a we find $b=0$.

Hence the equation of the line is $y={\cdot}39x$.

Conversely, whenever the equation of a line is given in the form

$$y=ax+b. \qquad \text{.................................(2)}$$

by giving to y the values 1, 2, 3..., corresponding values of x are obtained which may be plotted and the line obtained.

Plotting a Line.—In a similar manner to that of last example any two quantities which vary dependently on each other may be plotted on squared paper and an equation connecting the two determined.

The two forces applied to a machine called the *effort* and the *resistance* may be denoted by the letters E and R.

As R alters in value, E which depends on R also alters, and the "*Law of the machine*" as it is called may be written

$$E=aR+b.$$

It should be observed that if desirable A and B, or any two letters, may be used instead of E and R; this is in many cases very convenient, as, for example, when the variables are the *pressure* and *volume*, or *pressure* and *temperature of a gas*, p and v or p and t would be used instead of A or B or x and y.

Ex. 2. Let $y=x+2$ be a given equation.

When $x=0,\ y=2;\quad x=1,\ y=3;\quad x=2,\ y=4.$

Hence corresponding values of x and y may be tabulated in two columns, thus:

Values of x	0	1	2	3	4	5	6
Values of y	2	3	4	5	6	7	8

When $x=0$, $y=2$; hence as in Fig. 193 this gives one point in the line. When $x=1$, $y=3$; make a small dot or cross at the point of intersection of the lines 1 and 3. Again, when $x=2$, $y=4$; at the intersection of the lines through 2 and 4, make a cross or dot. Proceeding in this manner, any number of points lying on the line are obtained and the points joined by a fine line.

Conversely assuming that the line represents a series of plotted results of two variables A and B. To find the law connecting the two it is only necessary to substitute in the equation $A=aB+b$, the simultaneous known values of two points in the line; this will give two equations from which a and b can be determined as in the previous example.

As the values along *oy* represent values of A, and those along *ox* values of B; $\therefore$ from the line (Fig. 193)

when A is 3, B is 1,
when A is 8, B is 6.

Substituting these values in Eq. (1) and subtracting:

$$\begin{array}{rll}\therefore & 3= a\times 1+b & \ldots\ldots\ldots(1)\\ & 8= a\times 6+b & \ldots\ldots\ldots(2)\\ \hline & 5=5a & \therefore\ a=1.\end{array}$$

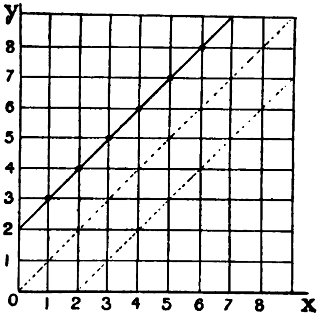


FIG. 193.—Plotting lines.

Hence $b=3-1=2$ from (1),
and the *law* is $A=B+2$.

The equation $y=x$,

$$\therefore \text{ when } x=0, \quad y=0,$$

is a line parallel to the preceding passing through the origin.

Again, when $y=x-2$,

$$\therefore \text{ when } x=0, \quad y=-2,$$

we have a parallel line intersecting the axis of y at a point -2, or 2 units below the origin.

The Simple Machines.—The so-called *simple* machines, the **lever, pulley, wheel and axle, inclined plane** and **screw,** or combinations of them, are considered as acted upon by two forces. Usually these are a comparatively small external force called the *effort*, and, in many cases, a much larger force consisting of a weight or load lifted, or resistance overcome. As the work done by a force is the product of the force multiplied by the distance through which it acts, we can arrange the mechanism in such a manner that the effort moves through a much greater distance than the weight is lifted, or the resistance overcome in the same period of time. The numerical magnitude of the effort is much less than the weight. Thus, if the effort E moves through 100 ft., whilst the weight W is raised 1 ft., then, neglecting friction, $W=100E$.

Mechanical Advantage.—The ratio of $\frac{W}{E}$ is called the *mechanical advantage of* the machine, but although of theoretical importance it is not so useful when applied to machines in which the frictional resistances in some cases may absorb as much as 60 or 70 per cent. of the effort expended.

To ascertain the *actual mechanical advantage* in a machine, a series of tests or experiments is made in the laboratory. The values of E and W are observed and tabulated; and the relation between these quantities can be found by plotting the observed values on squared paper and drawing a curve through the points. Usually these results are arranged so that a straight line can be drawn lying evenly among the points, and the equation of the line or the **Law of the Machine** can be obtained.

When the plotted results lie on a curve it would be difficult if not impossible, to express the relation between the variables in the nature of a law or equation connecting the two. In such cases a straight line may in some cases be obtained by plotting, instead of one of the variables, quantities derivable from it, such as the logarithms of the various values of one variable.

Ex. 1. If a cord or belt is passed over a rough cylinder, and forces T_2 and T_1 are applied at its extremities.

When the two forces T_2 and T_1 are equal, they balance. They continue to do so, as the load at T_2 is increased, until a certain value is reached and motion just begins.

The ratio of T_2 to T_1 when slipping occurs depends upon:

(*a*) The coefficient of friction (μ) between cord and cylinder.

(*b*) The angle θ embraced by the cord round the cylinder.

The variables are connected by the equation

$$\frac{T_2}{T_1}=e^{\mu\theta}, \text{ or } \log_e\frac{T_2}{T_1}=\mu\theta,$$

where T_2 and T_1 are the tensions in the cords on different sides of the cylinder; μ the coefficient of friction; θ the angle (expressed in circular measure) of contact of the cord with the cylinder; and e the base of the Naperian logarithms.

If $$\frac{T_2}{T_1}=k, \text{ then } \log_e k=\mu\theta \quad\text{.........................}(1)$$

In performing the experiment, T_1 may be kept constant, and T_2 is adjusted so that the cord just slips on the cylinder.

Ex. 2. If with a rope on a rough cylinder $T_1=8$ and $T_2=56$ when slipping occurs, find the value of T_2 when the angle of contact is 3π; the angle of contact in the first case being π.

Here $$k=7.$$

From Equation 1, $$\log_e 7=\mu\theta;\ \therefore\ 1{\cdot}9459=\mu\times\pi,$$

or $$\mu=\frac{1{\cdot}9459}{\pi}={\cdot}62.$$

When $$\theta=3\pi,\ \log_e k={\cdot}62\times3\pi,\ k=343{\cdot}5;$$
$$\therefore\ T_2=2404{\cdot}5 \text{ lbs.}$$

Common logarithms may be used instead of Napierian. The conversion from common (or logarithms to base 10) to Napierian is made by multiplying by 2·3026, p. 145.

EXERCISES. LXXXIII.

1. In experiments to determine the effort necessary to raise different loads with a crane, the following values were obtained. Plot the results on squared paper, and in each case find and write down the relation between the effort and the load.

E, the Effort in lbs.	60	70	79·5	89·5	99	108·5	118·25	128	138	147·5
R, the Weight or Load in lbs.	1083	1283	1483	1683	1883	2083	2283	2483	2683	2883

2. The annexed table gives a series of values of effort E, and load W, observed in testing a crane. Find the best average values of a and b in the approximate equation, $E=aW+b$.

W	28	56	84	112	140	168	196	224
E	14·2	26·6	38·1	50	59·1	72	81·8	91·5

3. The following results were obtained from an experiment on a screw-jack. Plot E and R on squared paper; find the relation between E and R and F and E.

Values of E	½	1	1½	2	2½	3	3½	4	4½	5	5½	6
R	10	19	29	46	51	66	78	89	101	113	123	135
F	29·5	56	83	106	136·5	159	187	212	237	262	289	315

4. In an experiment to determine the coefficient of friction between two surfaces (oak) the following values of F were required to give steady motion to a load W. Plot F and W as co-ordinates on squared paper, and find a mean value for μ where $\mu=\frac{F}{W}$.

Values of F	5	10	15	20	25	30	35	40
W	2	3	4½	6¼	7½	9½	10½	11¾

5. In the following examples a series of observed values of E and R is given. In each case they are known to follow a law approximately represented by $E=aR+b$, but there are errors of observation. Plot on squared paper, and determine the most probable values of a and b.

(i)

E	4	5	6	7	8	9	10	11	12	13
R	9½	12½	15½	18½	21½	24½	27¼	30½	33½	36½

(ii)

E	17	33	49	65	81	97	113	129	145	161
R	43	85	127	169	213	255	297	341	383	425

(iii)

E	5·75	8·5	11·25	14·25	17	20	22·5	25·25
R	7·56	11·56	15·56	19·56	23·56	27·56	31·56	35·5
F	·91	1·18	1·42	1·6	1·95	2·1	2·4	2·8

(iv)

E	·487	1·27	1·67	2·4	3·2	3·99	4·77	5·55	6·33
R	1·05	3·05	4·05	6·05	8·05	10·05	12·05	14·05	16·05

(v)

E	·5	1	1·5	2	2·5	3	3·5	4	4·5	5
R	7·5	18	28	38	49	59	68	80	90	100

(vi)

E	3	5·125	7·375	9·3125	11·5	14	16	18·25	21	23·25
R	7	14	21	28	35	42	49	56	63	70

(vii)

E	3	4·5	6	7·75	9·5	11	12·5	13·5	15
R	14	28	42	56	70	84	98	112	126

6. The following observed values of M and N are supposed to be related by a linear law, $M=aN+b$, but there are errors of observation. Find by plotting the values of M and N, the most probable values of a and b.

N	2·5	3·5	4·4	5 8	7·5	9·6	12·0	15·1	18·3
M	13·6	17·6	22 2	28·0	35·5	47·4	56·1	74·6	84·9

7. In an experiment the following results for θ and T_2 were obtained:

1	Values of θ, in circular measure	$\frac{\pi}{2}$	π	$\frac{3}{2}\pi$	2π	$2\frac{1}{2}\pi$	3π	$3\frac{1}{2}\pi$	4π
2	T_2	96	166	249	350	500	750	1180	1535
3	$\log T_2$	1·9823	2·2201	2·3962	2·5441	2·6990	2·8751	3·0719	3·1861
4	$\log \frac{T_2}{T_1}$	·2838	·5211	·6970	·8451	1·0000	1·1761	1·3729	1·4871

Plot columns 1 and 2 on squared paper; the points so plotted will lie on a curve. But if columns 1 and 3 and 1 and 4 are plotted, a straight line can be obtained. Find the equations of the line in the forms

$$\theta=a\log T_2+C,$$

$$\theta=a\log\frac{T_2}{T_1}+C.$$

8.

T_2	230	245	255	265	275	295	305	320
$\tan\theta$	·633	2·105	7·00	−·225	−·684	−2·077	−0	−·250

In the above experiment T_1 was 210 throughout, the observed values of T_2 and θ are given. Calculate and write down the values of $\log T_1$ and $\log T_2$. Plot on squared paper, and find the law as in the preceding example.

9. The following values of y and t were obtained in an experiment on viscosity of liquids:

(i)

y	180	164	149	137	125	115	105	97
t	0	39	119	115	198	239	279	319

(ii)

y	180	24	3	·7
t	0	14·4	28·4	42·2

(iii)

y	180	92	46	22	10	5
t	0	41·4	83·6	126·2	169·0	210·6

In each case plot t and log y on squared paper. Prove that

$$y=Ce^{-kt}.$$

Where C and k are constants, find the values of C and k in each case, and write down their values.

10. The following observed numbers are known to follow a law approximately represented by $y=ax+b$; but there are errors of observation. Find, by plotting the values on squared paper, the most probable values of a and b.

x	2	3	4·5	6	7	9	12	13
y	5·6	6·85	9·27	11·65	12·75	16·32	20·25	22·33

Plotting a Curve.—When an equation connecting two variables is given, then it has been seen that by giving to one of the variables a series of values, corresponding values of the other can be obtained.

Ex. 1. To plot a curve whose equation is $4y=x^2$;

$$\therefore\ y=\frac{x^2}{4}. \qquad (1)$$

By giving a series of values to x, 1, 2, 3, etc., we can obtain from Eq. (1) corresponding values of y.

Thus when $x=0,\ y=0$,

also when $x=1,\ y=\frac{1}{4}$.

It will be convenient to arrange the two sets of values of x and y as follows:

Values of x, - -	0	1	2	3	4	5	...	...	...
Corresponding values of y, - - -	0	¼	1	2¼	4	6¼	...	...	...

As y is 0 when x is 0, the curve passes through the origin (or point of intersection of the axes). Plotting the values of x and y from the two columns, as in Fig. 194, a series of points are obtained.

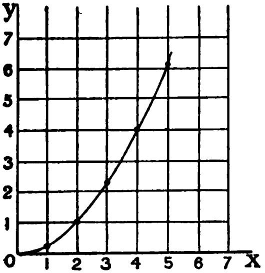


FIG. 194.—Graph of $4y=x^2$.

By bending a thin slip of wood a curve can be drawn as shown through the plotted points.

The curve plotted is of the form $y=ax^2$, and is known to be a **parabola.**

The constant a has been assumed to be positive, and equal to $\frac{1}{4}$.

By giving negative values to x, the numerical value of the constant remaining the same, the other half of the curve on the left of the line oy will be obtained. A series of values of x and y obtained from the equation $y=-\frac{1}{4}x^2$ is another parabola below the axis of x.

The space described by a body moving with uniform acceleration f is known to be given by the equation $s=\frac{1}{2}ft^2$.

Ex. 2. Plot the curve $s=\frac{1}{2}ft^2$; (*a*) assuming $f=\pm 4$, (*b*) $f=\pm 8$; (*c*) $f=\pm 32$ (values of t can be set off along ox, and corresponding values of s along oy).

Another important curve is the **rectangular hyperbola**; its equation is of the form $xy=c$.

Where c is a constant, let the equation be $xy=8$;

$$\therefore\ y=\frac{8}{x}\text{.................................}(1)$$

From (1), when x is 1, $y=8$. When x is $\frac{1}{2}$, $y=16$; when x is $\frac{1}{1000}$, $y=8000$, or, in other words, as the value of x is diminished, the corresponding value of y is increased.

It will be seen that when $x=0$, $y=\frac{8}{0}$, or is infinite in value. In other words, the curve gets nearer and nearer to the line oy as the value of x is diminished, but does not reach the axis at any finite distance from the origin. This is expressed by the symbols $y=\infty$ when $x=0$.

As equation (1) can be written $x=\frac{8}{y}$, it follows, as before, that when $y=0$, $x=\infty$.

The two lines or axes *ox* and *oy* are called **asymptotes,** and are said to meet (or touch) the curve at an infinite distance.

Arranging, as before, in two columns the series of values of x and y obtained from Eq. (1), we obtain :

Values of x,	0	1	2	3	4	5	6	7	8
Corresponding values of y,	∞	8	4	$\frac{8}{3}$	2	$\frac{8}{5}$	$\frac{8}{6}$	$\frac{8}{7}$	1

The curve, as already referred to, is of the utmost importance, it is shown in Fig. 195, and should be carefully plotted, assuming different values for the constant c.

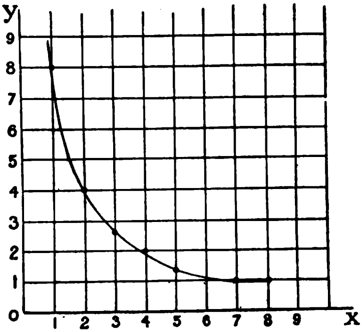


FIG. 195.—Hyperbola.

The rectangular hyperbola is the curve of expansion for a gas such as air, at constant temperature, and is often taken to represent the curve of expansion of superheated or saturated steam.

If p and v denote the *pressure* and *volume* respectively of a gas, the equation becomes $pv = \text{constant} = c$.

Equations.—The solution of an equation which would be difficult if not impossible, by algebraical methods, can be obtained by the use of squared paper.

Ex. 3. $x^2 + 3x^{\frac{3}{2}} - 2x - 15 = 0$. ....(1)

Denoting the value of the left-hand side by y, then for any assumed value of x a corresponding value of y can be found. Making these calculations, the numbers in the annexed table are obtained.

x	0	1	2	2·5	3	3·5
y	−15	−13	−6·516	−1·991	3·588	10·9

These numbers are plotted on squared paper (the values of y as ordinates), and a curve is drawn through the plotted points; at the point where the curve cuts the axis of x an approximate value of x is obtained, for which the left-hand side of Eq. (1) is zero, *i.e.* it is a solution of the equation. This value may be sufficiently accurate for all practical purposes; if not, having found that the required solution lies between $x=2\cdot5$ and $x=3$, we can calculate the values of y for values of x equal to 2·6, 2·7, etc.; again plot, and in this way obtain a solution to any desired degree of accuracy. If necessary to give confidence in the method it may be advisable to apply it to the solution of equations (quadratics), which can be solved by algebraical methods.

Equations are often met with in which the ratio of one variable quantity to another is required.

Ex. 4. Given $x^5=\frac{7\cdot4(x^6-y^6)}{x}$, find the value of $\frac{y}{x}$.

Multiplying both sides by x, then

$$x^6=7\cdot4x^6-7\cdot4y^6,\quad \frac{y}{x}=\sqrt[6]{\frac{64}{74}}=\cdot9761.$$

Maxima and Minima.—If a quantity varies in such a way that its value increases to a certain point and then diminishes, the decrease continuing until another point is reached, after which it begins to increase; then the former point is called a **maximum** and the latter a **minimum** value of the quantity.

Ex. 1. Given $2x^3-15x^2+24x+25=0$; determine the value of x so that the expression may be a maximum or a minimum.

Denoting by y the value of the left-hand side of the equation and substituting various values for x corresponding values of y are obtained, as shown in the annexed table.

x	0	·5	1	2	3	4	5
y	25	33·6	36	29	16	9	20

Plotting these values on squared paper and joining the points, the curve *ABCD* (Fig. 196) is obtained. From the tabulated values the expression seems to be a maximum when $x=1$, and a minimum when $x=4$; this is confirmed by the curve, the former value at *B*, the latter at *C*.

Also, as $y=2x^3-15x^2+24x+25$, $\frac{dy}{dx}=6x^2-30x+24$; this gives the *slope* at any point, or the angle which the tangent to the curve at any point makes with the axis of x. At the points B and C the angle is zero, *i.e.* the tangent is horizontal, hence $\frac{dy}{dx}=0$;

$$\therefore\ 6x^2-30x+24=0,$$

or x^3-5x+4

$$=(x-1)(x-4).$$

The required values are therefore $x=1$, $x=4$.

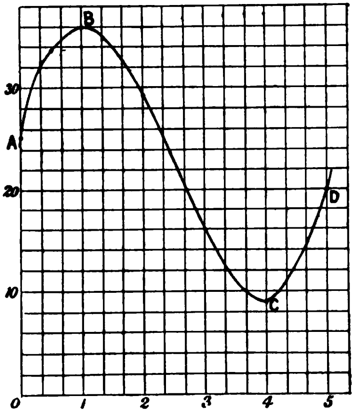


Fig. 196.—Curve showing maximum and minimum values.

Ex. 2. Divide the number 8 into two parts such that their product is a maximum.

Let x denote one part, then $8-x$ is the other; $x(8-x)$ or $8x-x^2$ is the product; denote this by y. Substitute values 0, 1, 2, 3, 4, 5, 6 for x, and find the corresponding values 0, 7, 12, 15, 16, 15, 12 for y. Plot on squared paper, and at $x=4$ a point corresponding to B (Fig. 196) is obtained; or let $y=8x-x^2$, then $\frac{dy}{dx}=8-2x$;

$$\therefore\ 8-2x=0, \text{ for a maximum, gives } x=4.$$

EXERCISES. LXXXIV.

Solve the equations:

1. $a^3=\frac{17(a^4-b^4)}{a}$. 2. $x^3=\frac{\cdot 6(x^4+y^4)}{x}$. 3. $y^4=\frac{\cdot 8(x^5+y^5)}{y}$.

4. $x^3-7x+6\cdot 9=0$. 5. $x^3-7x+7=0$.

6. Given $y=4x^3-x^2-2x+1$, find for what values of x the expression is a maximum or a minimum.

7. Divide the number 11·5 into two parts such that their product shall be a maximum.

8. Find the maximum or minimum values of

(i) $6x-x^2$, (ii) $1+3x-x^3$, (iii) $3x^4-16x^3+6x^2+72x-1$.

9. Divide the number n into two parts such that the product is a maximum.

10. If d is the diameter of a cylindrical tree, what is the breadth of the strongest rectangular beam which may be cut from it?

Harmonic Law.—Harmonic motion is a subject of great importance, not only in studying mechanics, but also in formulating theories of sound, light, heat, etc. The relation between x and y is given by the formula $y=A\sin(ax+b)$. When the values of the constants are given, or assumed, the relation between y and x can be set out in the form of what is called the sine curve. To commence with the simplest case, draw a semicircle with a radius 10 units, divide the quadrant into 9 equal parts, each denoting 10°, and from each point let fall a perpendicular; measure the length of each, and set out as the ordinates of a curve. Draw the curve AB, which represents one portion of a sine curve; the other portions of the sine curve can be obtained in like manner, but the construction is more easily carried out as follows: Let a point P move uniformly in a circle $ABA'B'$; draw the diameters AA', BB'; at any point P join P to C; also draw the perpendiculars PX and PY. The points X and Y are said to move with simple harmonic motions. If A is the radius of the circle, then $y=A\sin\theta$.

Ex. 1. Plot the curve $y=5\sin\left(\frac{\pi}{6}x+\frac{\pi}{3}\right)$.

Here when $x=0$,

$$y=5\sin\frac{\pi}{3}=5\sin 60^\circ=5\times{\cdot}866=4{\cdot}33.$$

When $x=1$,

$$y=5\sin(30^\circ+60^\circ)=5\sin 90^\circ=5.$$

When $x=2$,

$$y=5\sin\left(2\frac{\pi}{3}\right)=5\sin 120^\circ=5\times{\cdot}886=4{\cdot}33.$$

In this manner the following values are obtained. Plot as before, and draw curve.

x	0	1	2	3	4	5	6	7	8	9	10	11	12	13
y	4·33	5	4·33	2·5	0	−2·5	−4·33	−5	−4·33	−2·5	0	2·5	4·33	5

Ex. 2. Plot (or graph) the curve $y=4\sin\left(\frac{x}{1\cdot91}+\frac{\pi}{12}\right)$.

As before, a series of values are assumed for x, and corresponding values of y obtained.

Thus when $x=0$,

$$y=4\sin\left(\frac{\pi}{12}\right)=4\sin 15^\circ=1\cdot0352.$$

Similarly putting $x=1$, 2, etc., the following values of y are obtained:

x	0	1	2	3	4	5	6
y	1·0352	2·83	3·8686	3·8686	2·82	1·035	1·035

Ex. 3. Graph $y=\cdot8\sin\left(\frac{2x}{1\cdot91}\right)$.

As before the values of x and corresponding values of y are

x	0	1	2	3	4	5	6
y	0	0·6928	0·6928	0·0	0·6928	·6928	0

Using the given values (which should be verified) plot the curve, using the same scale and origin as before, and find the resultant curve obtained by adding the ordinate of one curve to the ordinate of the other.

Compound Interest Law.—This is expressed by the equation $y=Ae^{ax}$.

Ex. 1. To plot the curve $y=e^x$.

Where e is the base of the Napierian logarithms $=2\cdot7183$.

As before, give a series of values to x, and calculate y.

Thus when $x=0$, $y=e^0=1$.

$x=\cdot1$, $y=2\cdot7183^{\cdot1}=1\cdot1052$.

$x=\cdot2$, $y=2\cdot7183^{\cdot2}=1\cdot221$.

These values are obtained in the usual way by using logs. Thus when $x=\cdot2$, then

$$\log y=\cdot2\times\log 2\cdot7183$$
$$=\cdot2\times\cdot4343=\cdot08686.$$
$$\therefore\ \text{Antilog}\ \cdot0868=1\cdot221.$$

In a similar manner the other values can be calculated. The results are:

x	0	·1	·2	·3	·4	·5	·6	·7	·8	·9	1·0
y	1	1·105	1·221	1·350	1·492	1·649	1·822	2·014	2·225	2·460	2·7183

Plot on squared paper.

Plot $y=e^{-x}$; using the same values as before for x. Write down the corresponding values of y.

Catenary.—The curve assumed by a uniform flexible chain or cord when suspended from two points is called a **catenary**; its equation is

$$y=\frac{a}{2}\left(e^{\frac{x}{a}}+e^{-\frac{x}{a}}\right)$$

where e denotes the base of the Napierian logarithms and a is a constant.

Let $a=2$.

Then the equation is $y=\frac{2}{2}(e^x+e^{-x})=(e^x+e^{-x})$.

Let $x=0$, then $y=(e^0+e^{-0})=2$.

$x=·1$, then $y=(2·7183^{·1}+2·7183^{-·1})=(1·105+·905)=2·010$.

When $x=·2$, then $y=(2·7183^{·2}+2·7183^{-·2})=2·0402$.

Proceeding in this manner the following table is obtained. Plot on squared paper and draw the curve. When the constant a is made to assume different values, 1, 3, ..., the alteration in the curves obtained are very interesting.

x	0	·1	·2	·3	·4	·5	·6	·7	·8	·9	1·0
y	2	2·010	2·0402	2·090	2·162	2·255	2·371	2·511	2·674	2·866	3·086

EXERCISES. LXXXV.

Plot the curves represented by:

1. $y=\frac{x}{2}+2$. **2.** $y=\frac{x}{2}-1$. **3.** $y=3x-1$. **4.** $3y=4-x$.

5. $xy=1$. **6.** $xy=2x+3y-1$. **7.** $y=2x^2+1$.

8. $y=2x^{\frac{3}{2}}$. **9.** $x^2y=1$. **10.** $y\sqrt{x}=1$.

11. Plot the curves $pv=a$, when a is 100, 60, 20.

Plot the following:

12. $\dfrac{1-x^2}{1+x^2}=y.$ 13. $\dfrac{1+x^2}{1-x^2}=y.$ 14. $1-3x+4x^2=y.$

15. Find the least value of $y=x^2+3x+\frac{5}{2}$.

16. In the equation $y=0{\cdot}1e^x$ where e has the value 2·718, give values of ·1, ·2, ·3, etc., to x, and calculate the corresponding values of y. Plot the values of x and y on squared paper, and draw a curve through the points so obtained.

17. Plot the curves $y=\dfrac{x^3}{100}$ and $y=10e^{\cdot 1x}\sin\left(\dfrac{\pi}{6}x+\dfrac{\pi}{4}\right)$ between the values $x=0$ and $x=12$.

Slope of a Curve.—Taking two points P and Q on the curve, as shown in Fig. 197, the simultaneous values of P are

$$x=5,\quad y=1\tfrac{1}{2};$$

and of Q are

$$x=7,\quad y=5.$$

As the value of x increases from 5 to 7, the value of y increases from $1\frac{1}{2}$ to 5.

The increase in the value of x is called the *increment of x*, and the increase in the value of y is called the *increment of y*.

If the points P and Q be joined by a straight line cutting the curve, then by drawing QM and PM parallel to oy and ox respectively,

$$\text{increment of } y = QM,$$
$$\text{,, \quad ,, } x = PM;$$
$$\therefore \frac{\text{increment of } y}{\text{increment of } x}=\frac{QM}{PM}=\tan QPM=\frac{7}{4}.$$

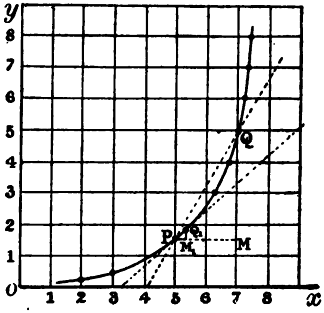


FIG. 197.—Slope of a curve.

If the point P be assumed to be fixed in position and the point Q to gradually move towards P, the increment of y and x will gradually get smaller and smaller.

When Q is indefinitely near to P the increments of y and x are indefinitely small.

The symbol dy is used to denote an indefinitely small increment of y; and the symbol dx an indefinitely small increment of x.

When Q is a point indefinitely near to P the line joining P and Q is a tangent to the curve at the point Q. Also the increments $\frac{QM}{PM}$ are represented by $\frac{dy}{dx}$. Thus $\frac{dy}{dx}$ denotes the limit of the ratio $\frac{dy}{dx}$ when both are indefinitely small, or the tangent of inclination of the curve OPQ at the point P to the axis ox.

Hence, to get the slope of a curve at a given point, it is not sufficient to join the point to another point on the curve, to produce the line and infer that the slope of the curve is the angle made by the line with the horizontal. To get the slope, the second point must be assumed to be brought closer and closer to the given point until they form two consecutive points on the curve indefinitely close together, then the line joining them is a tangent to the curve at the given point.

Area of diagram represents work done.—We may represent any forces which are doing work by vertical lines or ordinates and the space through which such forces act by horizontal distances or abscissae. Then joining the points so obtained we obtain a curve (Fig. 134), and the area included between it and the initial lines represents the total work done.

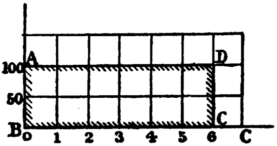


FIG. 198.—Area of diagram represents work done.

Thus, if AB (Fig. 198) denote the mean pressure on the piston of an engine, BC the length of stroke, or distance through which the piston moves, then work done in one stroke may be represented by the area of the rectangle $ABCD$.

Ex. 1. Let AB denote 100 lbs., and BC 6 ft., then work done

$$=100\times 6=600 \text{ ft.-lbs.}$$

When the force is variable, such as the force exerted in slowly compressing a spiral spring through a distance represented by BC (Fig. 199) the force increases from 0 at B to a maximum at C.

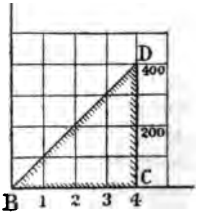


FIG. 199.—Work done in slowly compressing a spiral spring.

Ex. 2. Show by a diagram the amount of work done in *slowly compressing* a spiral spring through 4 inches, supposing the spring to shorten 1 inch for every 100 lbs. pressure.

Set off $BC=4$ inches to scale, the force required at C will obviously be 400 lbs., since by the data of the question 100 lbs. is required per inch compression. Hence making CD to scale to represent 400 lbs. and joining B to D; area of triangle BCD represents work done $=\frac{1}{2}(4\times 400)=800$ inch-lbs.

In instruments used for testing materials, a diagram may be drawn automatically, the abscissa corresponding on some scale to the space passed through, the ordinate of the curve at any point to the force exerted.

The area of the diagram between the base line and the curve represents the work done.

Thus in testing a piece of ductile material such as mild steel.

When a load of one ton per square inch is applied to the specimen it is found to alter its length. With double the load the alteration of length is doubled, and with half the load the alteration is one-half, and with three times the load three times the alteration in length is recorded; this is expressed by stating that *stress is directly proportional to strain*, which is known as Hooke's Law, or

$$\frac{\text{stress}}{\text{strain}}=\text{constant}=\textit{modulus of elasticity.}$$

The term *stress* indicates intensity, or

$$\textit{load on specimen} \div \textit{area of cross section,}$$

and $$\textit{strain} = \textit{alteration in length} \div \textit{original length.}$$

When the alteration in length is equal to the original length, or in other words, when the length is doubled the strain is unity, and the numerical value of the stress is then equal to the *modulus of elasticity.*

In india-rubber, spiral springs, etc., it may be easy and possible to obtain the strain unity, but this is practically impossible in iron, steel, etc.

A tensile test piece may be of the form shown in Fig. 200. It is cut from a plate or bar and afterwards machined so that the middle portion is parallel throughout. The ultimate tensile strength for steel plates would vary from 24 to 30 tons per sq. in. of section with an elongation of not less than 20 per cent. on a length of 8 inches.

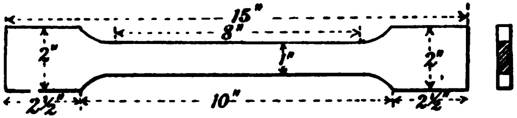


FIG. 200.—Tensile test-piece.

It is found that up to a certain point Hooke's law holds. If the loads are still further increased the strain is not proportional to stress but increases at a much more rapid rate. The specimen ultimately breaks at some point. Using the vertical line *oy* as an axis on which loads are measured, and *ox* as an axis on which the corresponding extensions are marked, we obtain a series of points. These, when carefully joined, give a curve; the area included between the curve and the base line at any point will give the work done up to that point, this area can be found by using a planimeter (p. 262) or by methods already described.

The work done up to the elastic limit is important and should be carefully noted; the stress in the material is called the *proof stress*; if P denote the force, and if x denote the alteration in length, work done $=\frac{P}{2}\times x$; this is called the *resilience*. Instead of loads and extensions, stresses and strains may be marked.

Ductility is the name given to the numerical value of the strain when fracture occurs.

Ex. 1. If a test piece, 8 inches long, is found to stretch 1½ inches when fracture occurs, the ductility is $\frac{1\frac{1}{2}}{8}=\frac{3}{16}$, or 18 per cent. approx.

Boyle's Law.—The curve (Fig. 195) is a graphic representation of Boyle's law; $pv=$constant, or *the volume of a given mass of gas is inversely proportional to the pressure upon it, as long as the temperature is kept constant.* A special case of $pv^n=$constant, when $n=1$.

Other curves are obtained when n is greater than unity.

Thus when $n = \frac{17}{16}$ or more accurately 1·0646 the curve obtained is the *saturation curve for steam.* When $n = 1{\cdot}408$ the curve becomes the *adiabatic curve* for a perfect gas like oxygen or nitrogen.

The curves just referred to are so much used both by physicists and engineers that any necessary amount of time spent in becoming familiar with them should be allowed.

The saturation curve of steam can be set out by obtaining from any set of tables the pressures and corresponding volumes of steam and using these as co-ordinates of points on a curve.

The curve of expansion of steam in a cylinder is only roughly part of a rectangular hyperbola (and in some cases the variation from this ideal form may be considerable). Yet the expansion curve of a good indicator diagram is found to be very approximately a hyperbola, and engineers consider this form the best that can be used to represent an ideal or hypothetical diagram.

It is said that in the endeavour to find an expression for the area of a hyperbolic curve Napier discovered logarithms, hence these are known as *Napierian* or **hyperbolic logarithms.**

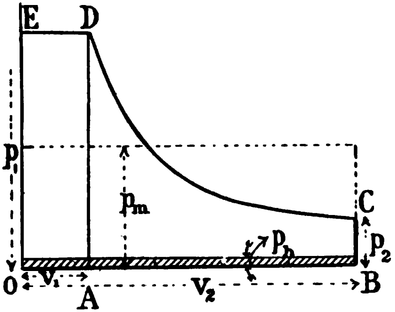


FIG. 201.—Area enclosed by hyperbolic curve.

Thus if *OEDCB* (Fig. 201) represent an area on the base *OB*, the portion *DC* being a portion of a rectangular hyperbola. If *DA* be drawn parallel to *OE*, then the area of the figure is the area of the rectangle *OEDA* together with the area of the curved portion *ADCB*.

Let $OE = p_1$, and $BC = p_2$. From Boyle's Law $p_1 V_1 = p_2 V_2$.

Hence area of rectangle $OEDA=p_1V_1$, area of curved portion $ADCB=p_1V_1\log_e\frac{V_2}{V_1}$.

Denoting $\frac{V_2}{V_1}$ by r; where r represents the ratio of expansion,

$$\text{area of figure}=p_1V_1+p_1V_1\log_e\frac{V_2}{V_1}=p_1V_1\left(1+\log_e\frac{V_2}{V_1}\right)$$

$$=p_1V_1(1+\log_e r)=p_2V_2(1+\log_e r)$$

But if p_m denotes the mean height of the diagram, or the mean pressure, then

$$\text{area of figure}=p_mV_2.$$

Hence
$$p_mV_2=p_2V_2(1+\log_e r);$$
$$\therefore\ p_m=V_2(1+\log_e r).$$

In many cases, due to what is called back pressure, a small area, length AB and height p_b (Fig. 197) must be subtracted from the mean pressure p_m.

Ex. 1. A cylinder is 20 inches diameter, the length of stroke 30 inches, and it makes 100 revolutions per minute; the initial pressure is 75 lbs. per sq. in.; cut off at $\frac{1}{5}$ the stroke; the back pressure 3 lbs.; assuming the expansion hyperbolic, find the horse power. ($\log_e 5=1{\cdot}61$.)

Here
$$p_m=\tfrac{75}{5}(1+1{\cdot}61)-3=36{\cdot}15.$$

$$\therefore\ \text{horse power}=\frac{36{\cdot}14\times2\frac{1}{2}\times10^2\times\pi\times200}{33000}=172{\cdot}1.$$

In all applications *absolute pressures* (and *not* pressures above that of the atmosphere) should be taken. Also, the law is only true for so-called *perfect gases* far removed from their point of liquefaction, and the temperature is assumed to remain constant during the changes of volume.

Ex. 2. Find the work done when 20 cub. ft. of air at an initial absolute pressure of 50 lbs. per sq. in. expands at constant temperature to a volume of 100 cub. ft.

Here
$$p_1=50\times144=7200\text{ lbs. per sq. ft.}$$

$$v_1=20,\quad v_2=100,\quad \frac{v_2}{v_1}=\frac{100}{20}=5;\ \log_e 5=1{\cdot}6094;$$

work done during expansion $=7200\times20\times1{\cdot}6094=231753{\cdot}6$ ft.-lbs.

Ex. 3. In experiments on a *spiral spring* the following values for load and corresponding *extensions* were observed:

Original length of spring 17·2 centimetres.

(*a*) Plot on squared paper the given values of extension and load and find the law connecting them; (*b*) also find E the modulus of elasticity.

First Spring.

Load on spring in pounds.	2	4	6	8	10	12	14
Extension in centimetres.	·60	·965	1·42	1·88	2·35	2·82	3·28

Setting the loads on the vertical axis and extensions on the horizontal axis, we obtain the mean line as in Fig. 202.

To obtain E; at one point we find a load of 10 lbs. produces an extension of 2·35 cm.

Hence an extension of 1 cm. would require $\frac{10}{2{\cdot}35}$;

$\therefore$ to double the length, *i.e.* a length 17·2 cm. requires

$$\frac{17{\cdot}2 \times 10}{2{\cdot}35}; \quad \therefore \quad E = 73{\cdot}2 \text{ lbs.}$$

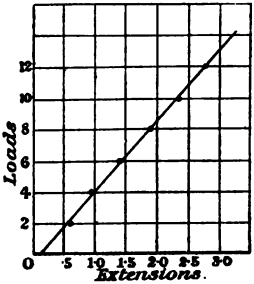


FIG. 202.

EXERCISES. LXXXVI.

1. Obtain the value of E for a 'spring' from the following table, original length of spring, 6·75 cm.

Load on spring in pounds.	2	4	6	8	10	12	14
Extension in centimetres.	1·095	2·175	3·25	4·335	5·42	6·51	7·6

2. Show by a diagram the amount of work done in *compressing* a spiral spring through 6 inches, supposing the spring to shorten 1 inch for every 100 lbs. pressure. If a weight of 100 lbs. *falls* from a height of 4 feet on the top of the spring, how much will it be compressed?

3. A spiral spring is stretched through 1 inch by a force of 10 lbs. Find the work done in stretching it through an additional length of 2 inches.

4. Find the number of foot-lbs. of work required to wind up a given chain 200 feet long, weighing 10 lbs. per foot length, which hangs by one end.

5. A chain 100 feet long and weighing 1 ton is hanging vertically. Draw a diagram of the work done in raising the lower end to meet the upper.

6. A chain weighing 2 lbs. per foot passes over a fixed smooth pulley, so that 14 feet hangs over on one side, and 6 feet on the other. Show by a diagram the work which will be done in pulling round the wheel until the upper end of the chain is one foot above the lower end. Write down its amount.

7. O, A, B are three points in order along a line OX. P is a force acting in the direction O to X, its magnitude varying inversely as the distance of its point of application from O; draw a diagram of the work done as the point of application of P moves from A to B, given $OA=1$, $OB=9$ feet, and the magnitude of P at $A=12$ lbs.

8. In testing a piece of round wire, diameter ·062 inches, length 5 ft.; the following values of the loads and corresponding elongations were observed:

Elongation,	56	70	84	98	112	126	140	151	162	182
Load, - -	·10	·11	·14	·17	·21	·23	·26	·295	·35	·395

Find area of cross-section of the wire. Also the stress and corresponding strain for each tabulated value; find also the mean value of E (modulus of elasticity). *Plot a stress-strain diagram.*

9. A round bar of iron 12 feet long and $1\frac{1}{2}$ square inches in sectional area is held at one end and pulled by a force till it stretches $\frac{1}{8}$ inch; find the force, the modulus of elasticity being 30,000,000.

10. Find the extension produced in a bar of wrought iron 4 feet long and 2 square inches in section by a suspended weight of $4\frac{5}{8}$ tons, the modulus of elasticity of the material being 20,000,000 pounds per square inch.

11. In an engine of 20 horse-power the piston speed is 300 feet per minute, initial pressure 50 lbs. per sq. in., back pressure 5 lbs. per sq. in. Cut off takes place at $\frac{1}{5}$ the stroke. Assuming the expansion to be hyperbolic, find the area of the piston of the engine. ($\log_e 5 = 1 \cdot 61$.)

12. If the initial pressure in a cylinder is 60 lbs. per sq. in., cut off at one-sixth the stroke; find the mean forward pressure.
($\log_e 6 = 1 \cdot 792$.)

13. If the initial pressure in a cylinder is 95 lbs. per sq. in., and cut off takes place at $\frac{1}{9}$ stroke, find the mean pressure.
($\log_e 9 = 2 \cdot 2$.)

CHAPTER XXI.

EVALUATION OF ALGEBRAIC AND TRIGONOMETRICAL FORMULAE.

Evaluation of Formulae.—In a design of even the simplest kind there must be some guiding principles on which it is based. Those who are merely content to copy the designs of others, or those who depend on some so-called 'rule of thumb,' often ignore this fact. But rough and ready methods do not always succeed, with the result that calculations, which should have been made at the outset, are after all found necessary. No process of guess and trial is really advantageous. The successful man must be able to accomplish satisfactorily the greatest amount of work in a given time, and this can only be effected by using every labour-saving device with which he is acquainted. Thorough study is thus true economy of time and labour. A few workmen may have the necessary experience and intuition to dispense with any form of calculation; but for most men calculations have to be made and formulae, such as can be found in the numerous "pocket books" or in the pages of technical papers, must be relied on. These formulae are found in many cases to present many difficulties to those who are not well acquainted with Mathematics.

To avoid to some extent this difficulty, the information need not necessarily be wrapped up in a formula; it may be presented in other forms. Thus, to take a simple case as an illustration. The relations between the pressure and volume or between the pressure and temperature of a quantity of steam are quantities of vital interest and importance to the engineer. These relations may be shown in at least three ways, viz., **tables,** by **curves,** or by **formulae.**

Comparing the three methods we find advantages and disadvantages in each case.

Tables.—If one of the three quantities—pressure, volume, or temperature of steam—is given, the other two can be found from tables similar in character to the portion given below. This is in some respects convenient, but as will readily be seen unless the 'tables' are very bulky only a comparatively few values of the pressures, volumes, and temperatures, can be given; and to find the volume and temperature corresponding to a given pressure not found in the tables, what is known as interpolation has to be resorted to.

Temperature.		Pressure. Lbs. per square inch.	Volume in cubic feet of 1 lb.
°C.	°F.		
160	320	89·86	4·816
165	329	101·9	4·280
170	338	115·1	3·814
175	347	129·8	3·410
180	356	145·8	3·057
185	365	163·3	2·748
190	374	182·4	2·476
195	383	203·3	2·236
200	392	225·9	2·025
205	401	250·3	1·838
210	410	276·9	1·672

Thus, *to find the temperature and volume of steam at a pressure of* 95 *lbs. per square inch.* Corresponding to a pressure 89·86 lbs. per sq. in. the temperature from the table is seen to be 320° F.; when the pressure is 101·9 lbs. per sq. in. the temperature is 329° F., or for a difference of $101·9-89·66$, or 12·24 lbs. per sq. in.; there is an increase of $329°-320°=9°$. Hence to find increase in temperature for an increase of $95·0-89·66$ or 5·34, we have $5·34\times9\div12·24=3·925$;

$$\therefore \text{ temperature is } 323·9° \text{ F.}$$

In a similar manner the volume can be obtained.

Curves.—This interpolation is not necessary when curves showing the relation between pressure and volume, and pressure and temperature, are drawn. These curves enable any intermediate value to be read off, but unless they are made to a comparatively large scale it will be impossible to obtain results with accuracy, and if made to a large scale the curves are not sufficiently portable and handy.

Formulae.—When the relations between two or more quantities can be expressed in a formula or in formulae; then, when the necessary data are given, the required quantities can be calculated. For instance, if p denote the pressure in lbs. per sq. in. of saturated steam, v the volume (in cubic ft.) of a pound; then the following formula being nearly true, viz.:

$$pv^{1\cdot065}=471\,;$$

we can, given the other, find either p or v.

Ex. 1. Given $p=95$, find v.

Here $95\times v^{1\cdot065}=471$ or $v^{1\cdot065}=\frac{471}{95}$.

$$\therefore\ 1\cdot065\log v=\log 471-\log 95=2\cdot6730-1\cdot9777=\cdot6953.$$

$$\therefore\ \log v=\frac{\cdot6953}{1\cdot065}=\cdot6529.\quad \text{Antilog. } \cdot6529=4496.\quad \therefore\ v=4\cdot496.$$

Other values for the index of v may be assumed and the value of p or v calculated. When the index is unity then $pv=a$, where a is some constant. Values have been obtained and a curve of this kind plotted (Fig. 195). Assuming the value of v to be 1·5 and a to be 100, the corresponding values of p can be obtained in the following cases:

(1) $pv^{1\cdot13}=100$; (2) $pv^{0\cdot9}=100$; (3) $pv=100$.

Thus let $v=1\cdot5$, find p.

From (1) $$p=\frac{100}{v^{1\cdot13}}\,;$$

$$\therefore\ \log p=\log 100-1\cdot13\log 1\cdot5=2\cdot0000-\cdot1990=1\cdot801.$$

$$\text{Antilog } \cdot801=6324\,;\ \therefore\ p=63\cdot24.$$

From (2) in like manner using the same value for v, we obtain $p=69\cdot42$, and from (3) $p=66\cdot66$.

It follows that, if the information cannot be expressed by a formula, either squared paper or tables must be used, though the use of a formula presents at once the most compact and portable form in which the information can be presented. The only disadvantage is that calculation must be employed. Unfortunately it sometimes happens that the mathematical symbols involved are sufficient to deter any but the most courageous of practical men. In the following pages an attempt is made to illustrate the evaluation of formulae, such as are continually required by engineers, builders, etc., and the data to which they refer is given in the majority of cases. It is impossible in the present pages to give the proofs of the rules used. For this information the student is referred to the numerous text-books in which they can be obtained. The mere fact that the proof cannot readily be obtained is no reason why any given formula shall, or shall not be used.

Bending.—In the preceding pages the bending moment at a section has been obtained. The *moment of inertia* of a section has been found also, both graphically and by calculation. In the consideration of the strength of materials these two quantities are connected by the equation $\frac{M}{I}=\frac{f}{y}$;

M denoting the bending moment, I the moment of inertia of the section, and f the stress (p. 347) in the material at a distance y from the neutral axis. Or

$$M=\frac{I}{y}f.$$

$\frac{I}{y}$ is called the **strength modulus**, and is usually denoted by Z.

$$\therefore\ M=fZ.$$

Values of Z for various sections are tabulated, or I and y, the value of Z can be calculated. In all cases, except where there is an applied tangential thrust or tension, the *neutral axis passes through the centre of gravity* of the section, and the value of y is the distance from the neutral axis to the extreme fibre of the material. Thus, in a wooden rectangular beam, denoting the breadth and depth by b and d respectively,

$$I=\tfrac{1}{12}bd^3,\ y=\frac{d}{2};\ \therefore\ Z=\frac{\tfrac{1}{12}bd^3}{\frac{d}{2}}=\frac{f}{6}bd^2 \ldots\ldots\ldots\ldots\ldots(1)$$

Ex. 2. If the load applied at the centre of a beam of oak 10 feet span be 6000 lbs., the depth 8 in., and safe stress 1500 lbs., find the breadth.

Here $$M=\frac{wl}{4}=\frac{6000\times10\times12}{4}=6000\times30\,;$$

$$\therefore\ \text{from (1)}\quad 6000\times30=\frac{1500}{6}\times b\times8^2,\ \therefore\ b=\frac{6000\times30\times6}{1500\times64}=11{\cdot}25\text{ in.}$$

Ex. 3. A stone cantilever supports a girder. The supported force being equal to 400 lbs. and acting at 16 in. from the wall; if the breadth be 10 in., find the depth. Safe stress 50 lbs. per sq. in.

$$M=400\times16,\ \text{or,}\ 400\times16=\frac{50}{6}\times10\times d^2\,;$$

$$\therefore\ d^2=\frac{400\times16\times6}{50\times10}=\frac{384}{5}\,;$$

$$\therefore\ \log d=\frac{1}{2}(\log384-\log5).\quad \text{Antilog}\ {\cdot}9426=8762\,;\ \therefore\ d=8{\cdot}76\text{ in.}$$

Ex. 4. The flanges of a cast-iron girder are 4″ × 1″ and 6″ × 1″, the depth 10″. If the span be 15 feet, find the load W which may be applied at the centre of the span. Safe stress compression 8 tons, tension $1\frac{1}{2}$ tons per sq. in.

The position of the neutral axis or centre of gravity is found to be at a distance of $4\frac{1}{2}''$ from bottom edge, and the moment of inertia $241\frac{1}{2}$ inch units.

Denoting the load by W, then

$$M=\frac{Wl}{4}=\frac{W\times15\times12}{4}=45W.$$

Also $$\frac{I}{y}f=\frac{241{\cdot}5}{4{\cdot}5}\times1\tfrac{1}{2}\,;$$

$$\therefore\ \text{from (1),}\quad W=\frac{241{\cdot}5\times1{\cdot}5}{4{\cdot}5\times45}=\frac{241{\cdot}5}{135}=1{\cdot}788\text{ tons.}$$

The weight of this girder having been obtained, it will be found a very useful exercise to find the load which may be safely applied at the centre of the beam when its own load has been taken into account.

Ex. 5. The flanges of a cast iron girder are 4″ × 1″ and 16″ × 1″ respectively. Find the uniformly distributed load per foot run (or per foot length) which the girder will carry. The depth of the girder is 20″, and the length 20′ 0″.

Let W denote the total load $= wl$ where w denotes the load per foot run.

Then $$M = \frac{Wl}{8} = \frac{W \times 20 \times 12}{8} = 30W.$$

In p. 324 I has been found to be 1950, and the neutral axis is at 7″ from bottom edge;

$$\therefore \; \frac{I}{y}f = \frac{1950}{7} \times 1\tfrac{1}{2}, \text{ or } 30W = \frac{1950 \times 3}{7 \times 2};$$

$$W = \frac{1950 \times 3}{14 \times 30} = \frac{195}{14} = 13{\cdot}93 \text{ tons};$$

$$\therefore \text{ load per foot run} = \frac{13{\cdot}93}{20} = {\cdot}696 = {\cdot}7 \text{ ton approximate.}$$

Rough Rule.—It is worth while to compare the result with that which would be obtained by using $\frac{Wl}{8} = Afd$. Where A denotes the area of either flange in square inches, d denotes the depth of the girder, and f the safe stress.

Here $$W = \frac{16 \times 1 \times 1\frac{1}{2} \times 20 \times 8}{20 \times 12} = 16 \text{ tons};$$

$$\therefore \; w = \frac{16}{20} = {\cdot}8 \text{ ton per foot.}$$

The rough rule gives too great a load as the result; hence, when it is employed, a lower value for the stress should be used. It is also necessary to consider the stiffness, or, in other words, to find the deflection of a beam.

Deflection.—If δ denote the deflection of length l when subjected to a load W;

then $$\delta = c\frac{Wl^3}{EI}, \qquad \text{.................................(1)}$$

where I denotes the moment of inertia of the cross-section and E the modulus of elasticity. It is not only necessary that a beam or girder shall be of sufficient strength, but it must also be sufficiently "stiff," *i.e.*, the deflection must not exceed a certain fixed amount. According to Tredgold this maximum is $\frac{1}{40}$″ per foot length.

Ex. 6. Find the deflection of the girder in the preceding example. Given $E = 1{\cdot}7 \times 10^7$. The value of c for a beam uniformly loaded

$$= \frac{5}{384}; \quad \therefore \; \delta = \frac{5}{384} \times \frac{14 \times 2240 \times (20 \times 12)^3}{1{\cdot}7 \times 10^7 \times 1950}.$$

Using logs as before we find $\delta=\frac{1}{8}$ in. But, according to Tredgold, the allowable deflection is $\frac{20}{40}=\frac{1}{2}$ in. for 20 ft. span. Hence, as it is only $\frac{1}{8}$ in. the beam is stiff enough.

Modulus of Elasticity.—The value of E is usually obtained for various materials by observing accurately the deflection due to a known load W, which may be applied at the centre of a span l; then I is known or can be calculated; δ, W, l, and the value of c (which is $\frac{1}{48}$) are known, and from (1) E can be obtained.

The deflection of a beam supported at each end and loaded at the centre is given by the formula

$$\delta=\frac{Wl^3}{48EI} \quad\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

To find the values of I and E in the following case.

Ex. 7. In an experiment the breadth and depth of a beam were respectively 1·02 inches and ·77 inch, the length 36 inches; and it was found that a load W of 16 lbs. applied at the middle of the beam produced a deflection of ·26 inch.

$$I=\frac{1}{12}\,.\,1{\cdot}02\times({\cdot}77)^3. \quad \text{Also from (1) } E=\frac{Wl^3}{48\,.\,\delta\,.\,I},$$

substituting the value of I;

$$\therefore\ \frac{16\times 36^3\times 12}{48\times{\cdot}26\times{\cdot}77^3\times 1{\cdot}02}, \quad \therefore\ E=1{\cdot}54\times 10^6.$$

Twisting Moment or Torque.—We have already found that the work done by a force is the product of the force and the space through which it acts (p. 290). In a similar manner if O is an axis (Fig. 203) and F denotes a force at a distance r from it. Then the work done $=F\,.\,s$ where s denotes the arc AB; but arc $AB=r\theta$ (p. 31);

$$\therefore\ \text{work done}=Fr\times\theta.$$

The product of F and r is called the **twisting moment** and is denoted by T, also θ for one revolution is 2π and for n revolutions $2\pi n$;

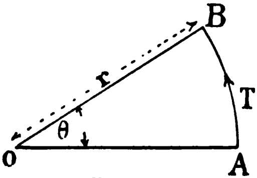


FIG. 203.

$$\therefore\ \text{horse-power}=\frac{2\pi nFr}{33000}=\frac{2\pi nT}{33000}. \quad\ldots\ldots\ldots\ldots(1)$$

If d denotes the diameter of a shaft, the relation between T and d (which is easily obtained by the simplest application of the calculus) is

$$T=\frac{\pi}{16}fd^3=\cdot196fd^3. \quad\text{.........................}(2)$$

If the shaft be hollow, the external and internal diameters being denoted by D and d respectively;

then $$T=\frac{\pi}{16}f\frac{D^4-d^4}{D}.$$

When a shaft is twisted, any line originally in the direction of the axis becomes displaced forming a spiral.

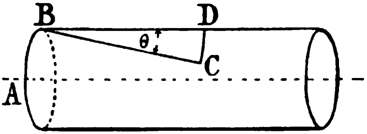


FIG. 204.

Thus, if the end A of a shaft (Fig. 204) be fixed and the line BD be displaced to BC,

Denoting BD by l and DBC by θ.

Then $$Gr\theta=fl,$$

or $$\theta=\frac{fl}{Gr}=\frac{2fl}{Gd} \quad\text{.......................}(3)$$

Where G is a constant called the modulus of transverse elasticity; from (2)

$$f=\frac{T}{\frac{\pi}{16}d^3}; \quad \therefore\ \theta=\frac{2Tl}{G\frac{\pi}{16}d^3}=\frac{32Tl}{\pi Gd^4}\text{................}(4)$$

Ex. 8. If the twisting moment is 20,000 inch-lbs., find the horse-power transmitted at 250 revolutions per minute.

From (1) $$\text{H.P.}=\frac{2\pi\times250\times20000}{33000\times12}=79\cdot3.$$

Ex. 9. Find the horse-power which can be transmitted by a shaft 3 inches in diameter when making 150 revolutions per minute; the shearing stress in the material not to exceed 7500 lbs. per square inch.

$$\text{H.P.}=\frac{\pi}{16}\times\frac{3^3\times7500\times2\pi\times150}{33000\times12}=94\cdot6.$$

Ex. 10. If a shaft 5 inches diameter be 12 feet long; the shear stress 9000 lbs. per square inch, and the modulus of transverse elasticity 9,000,000, find the angle of twist.

Here we have $\theta = \frac{2fl}{Gd} = \frac{2 \times 9000 \times 12 \times 12}{9000000 \times 5} = \frac{9 \times 288}{45000}$

$$\log \theta = \log 9 + \log 288 - \log 45000 ; \quad \therefore \quad \theta = {\cdot}0576 = 3^\circ {\cdot}89.$$

Combined Torsion and Bending.—When a shaft is subjected to simultaneous torsion and bending, the twisting moment *Te*, equivalent to both the twisting moment *T* and the bending moment *M* combined, is given by

$$Te = M + \sqrt{M^2 + T^2}.$$

Ex. 11. A wrought iron shaft is subjected simultaneously to a bending moment of 8,000 and a twisting moment of 15,000 in.-lbs. Find the twisting moment equivalent to these two, and the diameter of the shaft. Shearing stress 8000 lbs. per sq. in.

Draw a right-angled triangle (Fig. 205), its two perpendicular sides equal to T and M respectively, the hypothenuse $\sqrt{M^2 + T^2}$.

Describe arc of circle radius equal to M. Then

$$T_e = M + \sqrt{M^2 + T^2} = 25{,}000 \text{ in.-lbs.}$$

$$25000 = \frac{\pi}{16} 8000\, d^3,$$

from which $d = 2{\cdot}5$ inches.

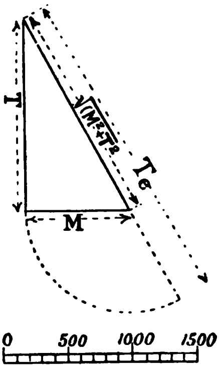


FIG. 205.

Tension and Bending.—When a bar is subjected to combined tension and bending, the stress due to each may be estimated, and the total stress obtained as in the following example:

Ex. 12. A steel boiler stay, 2 inches diameter and 20 feet long, supports an area equal to a square of 15 inches side, the steam pressure being 120 lbs. per sq. in. Calculate the greatest stress in the stay, taking into account both the direct pull and the bending due to the weight of the stay. Denoting the stress due to tension by f_t, and that due to bending by f_b.

Then $$f_t=\frac{15^2\times 120}{2^2\times\frac{\pi}{4}}=8592 \text{ lbs.}$$

Weight of stay $=2^2\times\pi\times 20\times 12\times \cdot 29=218\cdot 7$ lbs.

$$\therefore \frac{218\cdot 7\times 20\times 12}{8}=f_b\frac{\pi d^3}{32}.$$

$$\therefore f_b=\frac{218\cdot 7\times 20\times 12\times 32}{\pi\times 8\times 8}=8352 \text{ lbs.}$$

$\therefore$ Total stress $f=f_t+f_b=8592+8352=16944$ lbs. per sq. in.

Long Struts.—The breaking weight W (in tons) of a solid cast-iron column, when the length exceeds 30 times its diameter, may be obtained from Hodgkinson's formula,

$$W=44\cdot 06\frac{D^{3\cdot 55}}{L^{1\cdot 7}},$$

where L is the length of column in feet, and D is its diameter in inches.

Ex. 13. Let $L=16$ feet, $D=6$ inches.

Then $$W=44\cdot 06\frac{6^{3\cdot 55}}{16^{1\cdot 7}}.$$

$$\therefore \log W=\log 44\cdot 06+3\cdot 55\log 6-1\cdot 7\log 16$$
$$=1\cdot 6441+2\cdot 7626-2\cdot 0470=2\cdot 3597.$$

Antilog $\cdot 3597=2290$. $\therefore$ $W=229$ tons.

The safe load would be about $\frac{1}{10}$ the above, or 22·9 tons.

Euler's Formula.—$P=c\frac{\pi^2 EI}{l^2}$, where c is a constant depending upon the mode in which the ends are secured.

Ex. 14. If the maximum load on a piston is 163,072 lbs., find the diameter of the connecting rod, which is 10 ft. long.

Let $c=\frac{1}{5}$, $E=29{,}000{,}000$, $I=\frac{\pi}{64}d^4$.

$$\therefore 163072=\frac{\pi^2 E\frac{\pi}{64}d^4}{5\times(10\times 12)^2}.$$

$$\therefore d^4=\frac{163072\times 120^2\times 5\times 64}{\pi^3\times 29000000}; \quad \therefore d=5\cdot 4 \text{ inches.}$$

Flow of Water.—Prof. J. Thomson has determined the following rule: In a V-shaped notch, the sides of which make angles of 90° with each other, if Q denote the number of cubic feet of water flowing per second, then $Q=2\cdot634h^{\frac{5}{2}}$, where h (Fig. 206) is the depth of the water in feet.

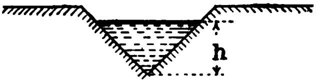


FIG. 206.

Ex. 15. Calculate the number of cubic feet per second and the number of gallons per hour passing over a V-shaped notch, when h is 2 feet.

$$Q=2\cdot634\times2^{\frac{5}{2}}.$$

$$\log Q=\log 2\cdot634\times\tfrac{5}{2}\log 2=1\cdot1732.$$

$$\text{Antilog. } \dot{1}732=149\,; \quad \therefore\ Q=14\cdot9.$$

Hence as there are $6\frac{1}{4}$ gallons in a cubic foot.

Number of gallons per hour $=14\cdot9\times6\frac{1}{4}\times3600=335{,}250$ gallons.

Direct-acting Engine.—When the mass m of the reciprocating parts and the length of stroke l of an engine are known, then, if we regard the connecting rod as very long, the inertia or inability to come to rest at the ends of the stroke is expressed by $P=m\omega^2l\div2$, where ω = angular velocity.

Ex. 16. The *weight* of the reciprocating parts of an engine is 300 lbs.; the revolutions, 80 per minute; and length of stroke, 3 ft.; find P.

$$\omega=\frac{2\pi\times80}{60}=\frac{8}{3}\pi\,;$$

$$P=\frac{300}{32\cdot2}\times\frac{8^2}{3^2}\times\pi^2\times\frac{3}{2}=981\text{ lbs.}$$

Boilers and Thick Cylinders.—If the thickness of a cylindrical shell is small compared with its diameter, such as obtains in a Lancashire or Cornish boiler, then the relation between the diameter d, the internal pressure p, thickness t, and safe or ultimate stress f, is obtained from

$$pd=2ft.$$

Ex. 17. Find thickness of the shell of a 7′ 6″ Lancashire boiler; pressure, 125 lbs. per square inch; $f=6$ tons per square inch; efficiency of joint, 70 per cent.

Here $$t=\frac{pd}{2f}\frac{70}{100}=\frac{125\times 90}{2\times 6\times 2240}\times\frac{100}{70}=\frac{5}{8}\text{ inch.}$$

In a thick cylinder, denoting the outer and inner diameters by D and d respectively,

$$\frac{D}{d}\text{ or }\frac{R}{r}=\sqrt{\frac{f+p}{f-p}}.$$

Ex. 18. The cylinder of an accumulator is 9 inches diameter, find the necessary thickness for a pressure of 700 lbs. $f=2100$ lbs.

$$\frac{R}{r}=\sqrt{\frac{2100+700}{2100-700}}\text{ gives }R=6{\cdot}34;$$

$\therefore$ thickness is 1·84 inches.

Average Velocity.—When a body of mass m is acted upon by a constant force, it produces an acceleration f, the numerical value of f being obtained from the equation $F=mf$.

Or, $$\therefore\ f=\frac{F}{m}\ \dots\dots\dots\dots(1)$$

The space described in a time t is given by

$$s=\tfrac{1}{2}ft^2.\ \dots\dots\dots\dots(2)$$

In both equations, if the uniform force is that due to gravity, the acceleration is denoted by g, and the space described is obtained from $s=\frac{1}{2}gt^2$.

We have already made use of this equation, and also the relation that $s=$(aver. vel.) × (time), or average velocity $=\dfrac{\text{space}}{\text{time}}$.

It remains to be seen that, to obtain a good approximation to the average velocity, we must consider the space described in a very small interval of time; these small increments are denoted by δs and δt respectively;

$$\therefore\ \text{average velocity}=\frac{\delta s}{\delta t}.$$

To clearly understand why the intervals or increments should be very small, it is best to find by actual substitution what sort of averages are obtained, when the time alters by a small amount.

Ex. 19. Suppose a body to fall from rest according to the law $s=16t^2$, find how far it falls between the times $t=1$ and $t=1{\cdot}1$; between $t=1$ and $t=1{\cdot}25$; between $t=1$ and $t=1{\cdot}5$;

$$s=16\times(1{\cdot}1)^2=19{\cdot}36 \text{ feet};$$

$$\therefore \text{ space in } {\cdot}1 \text{ second}=16\{(1{\cdot}1)^2-1^2\}=3{\cdot}36;$$

$$\therefore \text{ average velocity}=\frac{3{\cdot}36}{\frac{1}{10}}=33{\cdot}6.$$

Hence the actual velocity obtained is too great, and its inaccuracy becomes greater as the interval of time is increased.

Thus space in one-quarter of a second $=16\{(1\tfrac{1}{4})^2-1^2\}$

$$=16(\tfrac{25}{16}-1)=9;$$

$$\therefore \text{ actual velocity}=\frac{9}{{\cdot}25}=36.$$

If the interval be from 1 to $1{\cdot}01$,

$$\text{space}=16(1{\cdot}01^2-1^2)$$

$$=16(1{\cdot}0201-1)={\cdot}3216;$$

$$\therefore \text{ actual velocity}=\frac{{\cdot}3216}{{\cdot}01}=32{\cdot}16.$$

Other values for t may be assumed, the approximate value obtained becoming closer and closer to the actual value as the interval of time is diminished. Thus, the intervals of time may be ·001, ·0001 of a second, etc. These small intervals of time and corresponding small space described may be indicated in a convenient manner by the symbols δs and δt. An accurate value is obtained when the increments are made indefinitely small, and is indicated by $\frac{ds}{dt}$.*

---

* Let $s=\frac{1}{2}gt^2$ ....(1)

Denoting a small increment of s by δs and a small increment of t by δt, we obtain by substitution in (1)

$$s+\delta s=\tfrac{1}{2}g(t+\delta t)^2$$

$$=\tfrac{1}{2}g\{t^2+2t\delta t+(\delta t)^2\}. \qquad \text{.....................(2)}$$

Subtracting (1) from (2);

$$\therefore \delta s=gt\delta t+g(\delta t)^2,$$

or $$\frac{\delta s}{\delta t}=gt+g\delta t.$$

When the increment of time becomes indefinitely small the quantity $g\delta t$ ultimately vanishes, and we obtain

$$\frac{ds}{dt}=gt.$$

Ex. 20. Find how far from rest a body falls between the following times $t=5$ and $t=5{\cdot}1$, $t=5$ and $t=5{\cdot}01$, $t=5$ and $t=5{\cdot}001$; find the average velocity in each interval of time and the actual velocity when t is 5, $g=32{\cdot}2$,

$$s=16{\cdot}1\,(5{\cdot}1^2-5^2),\quad \therefore\ \frac{s}{t}=\frac{1{\cdot}01\times 16{\cdot}1}{\frac{1}{10}}=162{\cdot}6.$$

In a similar manner the average velocities in the remaining two cases can be found.

Actual velocity when t is 5 obtained from

$$\frac{ds}{dt}=v=gt=32{\cdot}2\times 5=161 \text{ ft. per sec.}$$

EXERCISES. LXXXVII.

1. Evaluate $a^{\frac{2}{3}}b\div\sqrt{c^2+\sin^3 A}$.

When $a=4{\cdot}268$, $b={\cdot}0249$, $c=3{\cdot}142$, $A=26°$.

2. Find to four significant figures the values of the following expressions:

$$\text{(i)}\ \log\left(\frac{2{\cdot}064}{48{\cdot}72}\right)^{\frac{5}{2}};$$

$$\text{(ii)}\ \sqrt{\sin 25°\,.\,\tan^3 230°};$$

and find what power of 2 is nearest to 10^8.

3. (i) $0{\cdot}25^{2{\cdot}19}$.

(ii) $\sqrt[3]{\sin 50°\tan^2 38°\,20'}$.

(iii) $2\log\frac{15}{8}-\log\frac{25}{162}+3\log\frac{4}{9}$.

4. (i) $\dfrac{(2{\cdot}625)^{\frac{5}{2}}\times{\cdot}0625}{16{\cdot}06^{\frac{1}{12}}}$.

(ii) $\dfrac{\sin 162°\tan^2 140°}{\sqrt[3]{\sec 105°}}$.

(iii) Solve the equation $2^x=9$.

5. Thickness of a pulley rim is given by the formula

$$t=0{\cdot}7\delta+{\cdot}005d.$$

Where δ = thickness of belt, d = diameter of pulley.

If thickness of belt $=\frac{3}{4}$-inch, diameter $=40$ inches, find t.

6. Find the value of x from the following:

$$x=\sqrt{\frac{(\frac{1}{4})^5\times 10{\cdot}144\times{\cdot}308}{250\times{\cdot}000414}}.$$

7. (i.) The moment of inertia I of a circle about a diameter is given by the formula $I=\frac{\pi d^4}{64}$. Find I when d has the following values: ·3724″, ·375″, 2″; also find Z (strength modulus) in each case.

(ii.) Calculate I from $\qquad I=\frac{bcWt^2}{4\pi^2}$.

Given $t=3{\cdot}087$, $W=\frac{1}{16}$lb., $b=19{\cdot}7$, $c=367{\cdot}5$.

8. Also, using the same values, calculate the value of V from

$$V=\frac{2\pi\delta I}{Wbct}\text{ where }\delta=9{\cdot}5.$$

9. Calculate the number of gallons of water flowing over a V-shaped notch per second, the height h being 4 ft.

10. If the value of G is obtained from the formula

$$G=\frac{cWlr^2}{\pi ka^4}.$$

Find the value of G when $l=25\pi$, $r=1{\cdot}25$, $k={\cdot}0282$, $a={\cdot}25$, and $c=32$.

11. Also when l and r are unaltered, but $k={\cdot}0223$, $c={\cdot}14058^{-1}$.

12. From the formula $pv^{1{\cdot}065}=471$. Given $p=90$, find the value of v.

13. From the formula $pv^{1{\cdot}13}=100$. Calculate p when $v=2$, 2·5, 3, 3·5, 4.

14. Using the formula $pv^{0{\cdot}9}=100$, find p when v has the same values as before.

15. From $pv=100$ calculate p when v has the values 2, 2·5, 3, 3·5, 4.

16. If the law connecting the pressure and volume of steam be $pv^{1{\cdot}0646}=479$. For the following values of p calculate the corresponding values of v: 80, 90, 100, 120, 140, 180, 200, 220, 280.

Plot $\frac{1}{v}$ and p on squared paper, and show that the relation between them may be expressed in the form $\frac{1}{v}=ap+c$; hence, find the values of a and c.

17. If $pv^n=475$, calculate the value of v when p is 100, and when n is (a) $\frac{17}{16}$, (b) $\frac{10}{9}$, (c) $\frac{14}{10}$.

18. From the formula $P=m\omega^3l$, calculate the value of P in the following:

(i) $m=350$ lbs.; stroke 2 ft., 300 revs. per min.
(ii) $m=400$ lbs.; stroke $1\frac{2}{3}$ ft., 200 revs. per min.
(iii) $m=470$ lbs.; stroke 2 ft., 200 revs. per min.

19. Find the twisting moment when a shaft transmits 20 H.P. at 90 revolutions per minute.

20. If the angle of torsion in a shaft is not to exceed 1° per 3 ft. length, and stress not greater than 8000 lbs. per sq. in.; find the diameter of shaft to satisfy both conditions.

21. In a wrought iron shaft $M = 10000$ in.-lbs., $T = 20000$ in.-lbs., if $f = 7800$; find Te and d.

22. If the values of T and M be 1333500 and 240000 in.-lbs. respectively and $f = 8000$ lbs. per sq. in.; find Te and d.

23. Find the diameter of a mild steel shaft when $T = 480000$ in.-lbs., $f = 9000$; also find the increase in the diameter if there is in addition a bending moment M of 360000 in.-lbs.

24. Find the diameter of a shaft for 100 H.P. at 50 revolutions per min., the angle of torsion 1° per 10 ft. length, $G = 10000000$.

25. Calculate the value of d the diameter of a pipe obtained from the two equations;

$$d^{5\cdot 1} = \cdot 35\left(\frac{154\cdot 74}{\frac{\pi}{4}}\right)^2 \text{ and } d^5 = \cdot 0006\left(\frac{154\cdot 74}{\frac{\pi}{4}}\right)^2 \times 500.$$

26. Calculate by Hodgkinson's formula the safe loads for solid cast-iron columns of length 16 feet, the diameters being 5, 8, and 10 inches respectively.

27. Find the value of d from the formula $d = \cdot 01363\sqrt{(l\sqrt{P} + \cdot 79P)}$ where $P = 163072$, $l = 120$ ins.

28. The diameter of a piston rod may be obtained from

$$d = \cdot 0324\sqrt[4]{Pl^2}.$$

Find d when $P = 163072$, and $l = 60''$.

29. The work absorbed per revolution by the friction of a shaft in its bearing is given by $2\pi rs \sin \alpha$. Given $s = 20$ tons, $r = 9''$, $\mu = \tan \alpha = \cdot 1$; find work absorbed in one revolution.

30. The diameter of a steel rod is ·3725 inches, when supported at each end the distance apart of the supports being 19″, and a load of 14 lbs. applied at the centre, the deflection δ was found to be ·07036″. From the formula $E = \dfrac{Wl^3}{48\delta I}$ find E.

31. In a hydraulic main if the internal diameter be 6″; internal pressure, 700 lbs. per sq. in.; $f = 3000$ lbs.; find t the thickness; also find t when $p = 1400$ lbs.

32. In a boiler, 6 ft. diameter, the plates are $\frac{1}{2}$ in. thick, if efficiency of joint is 65 %, and safe stress 11,000 lbs.; find safe pressure of steam.

33. Find the diameter of a shaft to transmit 100 H.P. at 50 revolutions per minute; angle of torsion, 1° per 10 ft. length; $G = 10^6$.

34. A body of 200 lbs. is acted on by a force F which alters. No other force acts on the body. When the body has passed through the distance x feet, the force in pounds is as follows:

x	0	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9	1·0
F	20	21	21	20	19	18·5	18·0	13·5	9	4·5	0

Using either a graphical or arithmetical method, find:

(*a*) The average force acting on the body through the total distance of 1 foot.

(*b*) The work done upon the body from $x=0$ to $x=·4$.

(*c*) The answer to (*b*) being the kinetic energy added to the body; if the velocity was 0 when x is 0, what is the velocity when $x=·4$?

35. Suppose a body to have fallen h feet in t seconds from rest according to the law $h=16·1t^2$; find how far it falls between the times $t=3$ and $t=3·1$; between $t=3$ and $t=3·01$; between $t=3$ and $t=3·001$. Find the average velocity in each of these intervals of time. What is meant by the actual velocity when t is 3 seconds?

36. What is meant by the slope of a curve at a given point on it? How is it measured?

Suppose that two curves have been plotted such that in the one the abscissae and ordinates represent *time* and *distance*, while in the other they represent, say, the *temperature* and *pressure* of a given volume of gas: what will be represented by the slope? Obtain an expression for it in the former case if the distance s and time t are connected by the equation

$$s=5t+2·1t^2.$$

Give the numerical value at the instant when $t=5$.

MATHEMATICAL TABLES.

TABLE I. CONVENIENT NUMBERS.

$\pi = 3{\cdot}1416 = 3{\cdot}142$ or $\frac{22}{7}$.
$\pi^2 = 9{\cdot}87$.
$g = 32{\cdot}2$.

$\log \pi = {\cdot}4972$.
,, $2{\cdot}718 = {\cdot}4343$.
,, ${\cdot}7854 = \bar{1}{\cdot}8951$.
,, $62{\cdot}3 = 1{\cdot}7945$.

TABLE II. USEFUL CONSTANTS.

1 inch = 2·54 centimetres.
1 mile = 5280 ft. = 1760 yards.
1 cubic foot of water = 62·3 lbs.
1 gallon of water = 10 lbs. = 277·27 cub. in.
10 links = 1 linear chain = 22 yards.
10 sq. chains = 1 acre = 4840 sq. yards.
1 horse-power = 33,000 ft.-lbs. per min. = 746 watts.
Volts × amperes = watts.
1 atmosphere = 14·7 lbs. per sq. in.; corresponds to the pressure due to a column of water 34 ft. high, or a column of mercury 760 mm.
1 radian = 57·3 degrees.
The base of Napierian logarithms is $e = 2{\cdot}7183$.
To convert common into Napierian logarithms multiply by 2·3026.

TABLE III. RELATIVE WEIGHTS.

NAME.	Weight of Unit Volume in pounds.		Relative Density, or Specific Gravity.
	Cub. ft.	Cub. in.	
Water,	62·3	·036	1
Cast Iron,	450	·26	7·2
Wrought Iron,	480	·28	7·698
Steel,	490	·29	7·85
Brass,	515	·298	8·2
Copper,	552	·3192	8·9
Lead,	712	·4121	11·418

TABLE IV. LOGARITHMS.

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374	4	8	12	17	21	25	29	33	37
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755	4	8	11	15	19	23	26	30	34
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106	3	7	10	14	17	21	24	28	31
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430	3	6	10	13	16	19	23	26	29
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732	3	6	9	12	15	18	21	24	27
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014	3	6	8	11	14	17	20	22	25
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279	3	5	8	11	13	16	18	21	24
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529	2	5	7	10	12	15	17	20	22
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765	2	5	7	9	12	14	16	19	21
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989	2	4	7	9	11	13	16	18	20
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	2	4	6	8	11	13	15	17	19
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	2	4	6	8	10	12	14	16	18
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	2	4	6	8	10	12	14	15	17
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	2	4	6	7	9	11	13	15	17
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	2	4	5	7	9	11	12	14	16
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	2	3	5	7	9	10	12	14	15
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	2	3	5	7	8	10	11	13	15
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	2	3	5	6	8	9	11	13	14
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	2	3	5	6	8	9	11	12	14
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	1	3	4	6	7	9	10	12	13
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	1	3	4	6	7	9	10	11	13
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	1	3	4	6	7	8	10	11	12
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	1	3	4	5	7	8	9	11	12
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	1	3	4	5	6	8	9	10	12
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	1	3	4	5	6	8	9	10	11
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	1	2	4	5	6	7	9	10	11
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	1	2	4	5	6	7	8	10	11
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	1	2	3	5	6	7	8	9	10
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	1	2	3	5	6	7	8	9	10
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	1	2	3	4	5	7	8	9	10
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	1	2	3	4	5	6	8	9	10
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	1	2	3	4	5	6	7	8	9
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	1	2	3	4	5	6	7	8	9
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	1	2	3	4	5	6	7	8	9
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	1	2	3	4	5	6	7	8	9
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	1	2	3	4	5	6	7	8	9
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	1	2	3	4	5	6	7	7	8
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	1	2	3	4	5	5	6	7	8
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	1	2	3	4	4	5	6	7	8
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	1	2	3	4	4	5	6	7	8
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	1	2	3	3	4	5	6	7	8
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	1	2	3	3	4	5	6	7	8
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	1	2	2	3	4	5	6	7	7
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	1	2	2	3	4	5	6	6	7
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	1	2	2	3	4	5	6	6	7

TABLE IV. LOGARITHMS.

	0	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	1	2	2	3	4	5	5	6	7
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	1	2	2	3	4	5	5	6	7
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	1	2	2	3	4	5	5	6	7
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	1	1	2	3	4	4	5	6	7
69	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	1	1	2	3	4	4	5	6	7
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	1	1	2	3	4	4	5	6	6
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	1	1	2	3	4	4	5	6	6
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	1	1	2	3	3	4	5	6	6
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	1	1	2	3	3	4	5	5	6
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	1	1	2	3	3	4	5	5	6
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	1	1	2	3	3	4	5	5	6
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	1	1	2	3	3	4	5	5	6
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	1	1	2	3	3	4	5	5	6
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	1	1	2	3	3	4	4	5	6
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	1	1	2	2	3	4	4	5	6
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	1	1	2	2	3	4	4	5	6
71	8513	8519	8525	8531	8537	8543	8459	8555	8561	8567	1	1	2	2	3	4	4	5	5
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	1	1	2	2	3	4	4	5	5
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	1	1	2	2	3	4	4	5	5
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	1	1	2	2	3	4	4	5	5
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	1	1	2	2	3	3	4	5	5
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	1	1	2	2	3	3	4	5	5
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	1	1	2	2	3	3	4	4	5
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	1	1	2	2	3	3	4	4	5
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	1	1	2	2	3	3	4	4	5
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	1	1	2	2	3	3	4	4	5
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	1	1	2	2	3	3	4	4	5
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	1	1	2	2	3	3	4	4	5
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	1	1	2	2	3	3	4	4	5
84	9243	9248	9253	9258	9263	9269	9274	7279	9284	9289	1	1	2	2	3	3	4	4	5
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	1	1	2	2	3	3	4	4	5
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	1	1	2	2	3	3	4	4	5
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	0	1	1	2	2	3	3	4	4
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	0	1	1	2	2	3	3	4	4
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	0	1	1	2	2	3	3	4	4
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	0	1	1	2	2	3	3	4	4
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633	0	1	1	2	2	3	3	4	4
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	0	1	1	2	2	3	3	4	4
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	0	1	1	2	2	3	3	4	4
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	0	1	1	2	2	3	3	4	4
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	0	1	1	2	2	3	3	4	4
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	0	1	1	2	2	3	3	4	4
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	0	1	1	2	2	3	3	4	4
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	0	1	1	2	2	3	3	4	4
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	0	1	1	2	2	3	3	3	4

TABLE V. ANTILOGARITHMS.

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
·00	1000	1002	1005	1007	1009	1012	1014	1016	1019	1021	0	0	1	1	1	1	2	2	2
·01	1023	1026	1028	1030	1033	1035	1038	1040	1042	1045	0	0	1	1	1	1	2	2	2
·02	1047	1050	1052	1054	1057	1059	1062	1064	1067	1069	0	0	1	1	1	1	2	2	2
·03	1072	1074	1076	1079	1081	1084	1086	1089	1091	1094	0	0	1	1	1	1	2	2	2
·04	1096	1099	1102	1104	1107	1109	1112	1114	1117	1119	0	1	1	1	1	2	2	2	2
·05	1122	1125	1127	1130	1132	1135	1138	1140	1143	1146	0	1	1	1	1	2	2	2	2
·06	1148	1151	1153	1156	1159	1161	1164	1167	1169	1172	0	1	1	1	1	2	2	2	2
·07	1175	1178	1180	1183	1186	1189	1191	1194	1197	1199	0	1	1	1	1	2	2	2	2
·08	1202	1205	1208	1211	1213	1216	1219	1222	1225	1227	0	1	1	1	1	2	2	2	3
·09	1230	1233	1236	1239	1242	1245	1247	1250	1253	1256	0	1	1	1	1	2	2	2	3
·10	1259	1262	1265	1268	1271	1274	1276	1279	1282	1285	0	1	1	1	1	2	2	2	3
·11	1288	1291	1294	1297	1300	1303	1306	1309	1312	1315	0	1	1	1	2	2	2	2	3
·12	1318	1321	1324	1327	1330	1334	1337	1340	1343	1346	0	1	1	1	2	2	2	2	3
·13	1349	1352	1355	1358	1361	1365	1368	1371	1374	1377	0	1	1	1	2	2	2	3	3
·14	1380	1384	1387	1390	1393	1396	1400	1403	1406	1409	0	1	1	1	2	2	2	3	3
·15	1413	1416	1419	1422	1426	1429	1432	1435	1439	1442	0	1	1	1	2	2	2	3	3
·16	1445	1449	1452	1455	1459	1462	1466	1469	1472	1476	0	1	1	1	2	2	2	3	3
·17	1479	1483	1486	1489	1493	1496	1500	1503	1507	1510	0	1	1	1	2	2	2	3	3
·18	1514	1517	1521	1524	1528	1531	1535	1538	1542	1545	0	1	1	1	2	2	2	3	3
·19	1549	1552	1556	1560	1563	1567	1570	1574	1578	1581	0	1	1	1	2	2	3	3	3
·20	1585	1589	1592	1596	1600	1603	1607	1611	1614	1618	0	1	1	1	2	2	3	3	3
·21	1622	1626	1629	1633	1637	1641	1644	1648	1652	1656	0	1	1	2	2	2	3	3	3
·22	1660	1663	1667	1671	1675	1679	1683	1687	1690	1694	0	1	1	2	2	2	3	3	3
·23	1698	1702	1706	1710	1714	1718	1722	1726	1730	1734	0	1	1	2	2	2	3	3	4
·24	1738	1742	1746	1750	1754	1758	1762	1766	1770	1774	0	1	1	2	2	2	3	3	4
·25	1778	1782	1786	1791	1795	1799	1803	1807	1811	1816	0	1	1	2	2	2	3	3	4
·26	1820	1824	1828	1832	1837	1841	1845	1849	1854	1858	0	1	1	2	2	3	3	3	4
·27	1862	1866	1871	1875	1879	1884	1888	1892	1897	1901	0	1	1	2	2	3	3	3	4
·28	1905	1910	1914	1919	1923	1928	1932	1936	1941	1945	0	1	1	2	2	3	3	4	4
·29	1950	1954	1959	1963	1968	1972	1977	1982	1986	1991	0	1	1	2	2	3	3	4	4
·30	1995	2000	2004	2009	2014	2018	2023	2028	2032	2037	0	1	1	2	2	3	3	4	4
·31	2042	2046	2051	2056	2061	2065	2070	2075	2080	2084	0	1	1	2	2	3	3	4	4
·32	2089	2094	2099	2104	2109	2113	2118	2123	2128	2133	0	1	1	2	2	3	3	4	4
·33	2138	2143	2148	2153	2158	2163	2168	2173	2178	2183	0	1	1	2	2	3	3	4	4
·34	2188	2193	2198	2203	2208	2213	2218	2223	2228	2234	1	1	2	2	3	3	4	4	5
·35	2239	2244	2249	2254	2259	2265	2270	2275	2280	2286	1	1	2	2	3	3	4	4	5
·36	2291	2296	2301	2307	2312	2317	2323	2328	2333	2339	1	1	2	2	3	3	4	4	5
·37	2344	2350	2355	2360	2366	2371	2377	2382	2388	2393	1	1	2	2	3	3	4	4	5
·38	2399	2404	2410	2415	2421	2427	2432	2438	2443	2449	1	1	2	2	3	3	4	4	5
·39	2455	2460	2466	2472	2477	2483	2489	2495	2500	2506	1	1	2	2	3	3	4	5	5
·40	2512	2518	2523	2529	2535	2541	2547	2553	2559	2564	1	1	2	2	3	4	4	5	5
·41	2570	2576	2582	2588	2594	2600	2606	2612	2618	2624	1	1	2	2	3	4	4	5	5
·42	2630	2636	2642	2649	2655	2661	2667	2673	2679	2685	1	1	2	2	3	4	4	5	6
·43	2692	2698	2704	2710	2716	2723	2729	2735	2742	2748	1	1	2	3	3	4	4	5	6
·44	2754	2761	2767	2773	2780	2786	2793	2799	2805	2812	1	1	2	3	3	4	4	5	6
·45	2818	2825	2831	2838	2844	2851	2858	2864	2871	2877	1	1	2	3	3	4	5	5	6
·46	2884	2891	2897	2904	2911	2917	2924	2931	2938	2944	1	1	2	3	3	4	5	5	6
·47	2951	2958	2965	2972	2979	2985	2992	2999	3006	3013	1	1	2	3	3	4	5	5	6
·48	3920	3027	3034	3041	3048	3055	3062	3069	3076	3083	1	1	2	3	4	4	5	6	6
·49	3090	3097	3105	3112	3119	3126	3133	3141	3148	3155	1	1	2	3	4	4	5	6	6

TABLE V. ANTILOGARITHMS.

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
·50	3162	3170	3177	3184	3192	3199	3206	3214	3221	3228	1	1	2	3	4	4	5	6	7
·51	3236	3243	3251	3258	3266	3273	3281	3289	3296	3304	1	2	2	3	4	5	5	6	7
·52	3311	3319	3327	3334	3342	3350	3357	3365	3373	3381	1	2	2	3	4	5	5	6	7
·53	3388	3396	3404	3412	3420	3428	3436	3443	3451	3459	1	2	2	3	4	5	6	6	7
·54	3467	3475	3483	3491	3499	3508	3516	3524	3532	3540	1	2	2	3	4	5	6	6	7
·55	3548	3556	3565	3573	3581	3589	3597	3606	3614	3622	1	2	2	3	4	5	6	7	7
·56	3631	3639	3648	3656	3664	3673	3681	3690	3698	3707	1	2	3	3	4	5	6	7	8
·57	3715	3724	3733	3741	3750	3758	3767	3776	3784	3793	1	2	3	3	4	5	6	7	8
·58	3802	3811	3819	3828	3837	3846	3855	3864	3873	3882	1	2	3	4	4	5	6	7	8
·59	3890	3899	3908	3917	3926	3936	3945	3954	3963	3972	1	2	3	4	5	5	6	7	8
·60	3981	3990	3999	4009	4018	4027	4036	4046	4055	4064	1	2	3	4	5	6	6	7	8
·61	4074	4083	4093	4102	4111	4121	4130	4140	4150	4159	1	2	3	4	5	6	7	8	9
·62	4169	4178	4188	4198	4207	4217	4227	4236	4246	4256	1	2	3	4	5	6	7	8	9
·63	4266	4276	4285	4295	4305	4315	4325	4335	4345	4355	1	2	3	4	5	6	7	8	9
·64	4365	4375	4385	4395	4406	4416	4426	4436	4446	4457	1	2	3	4	5	6	7	8	9
·65	4467	4477	4487	4498	4508	4519	4529	4539	4550	4560	1	2	3	4	5	6	7	8	9
·66	4571	4581	4592	4603	4613	4624	4634	4645	4656	4667	1	2	3	4	5	6	7	9	10
·67	4677	4688	4699	4710	4721	4732	4742	4753	4764	4775	1	2	3	4	5	7	8	9	10
·68	4786	4797	4808	4819	4831	4842	4853	4864	4875	4887	1	2	3	4	6	7	8	9	10
·69	4898	4909	4920	4932	4943	4955	4966	4977	4989	5000	1	2	3	5	6	7	8	9	10
·70	5012	5023	5035	5047	5058	5070	5082	5093	5105	5117	1	2	4	5	6	7	8	9	11
·71	5129	5140	5152	5164	5176	5188	5200	5212	5224	5236	1	2	4	5	6	7	8	10	11
·72	5248	5260	5272	5284	5297	5309	5321	5333	5346	5358	1	2	4	5	6	7	9	10	11
·73	5370	5383	5395	5408	5420	5433	5445	5458	5470	5483	1	3	4	5	6	8	9	10	11
·74	5495	5508	5521	5534	5546	5559	5572	5585	5598	5610	1	3	4	5	6	8	9	10	12
·75	5623	5636	5649	5662	5675	5689	5702	5715	5728	5741	1	3	4	5	7	8	9	10	12
·76	5754	5768	5781	5794	5808	5821	5834	5848	5861	5875	1	3	4	5	7	8	9	11	12
·77	5888	5902	5916	5929	5943	5957	5970	5984	5998	6012	1	3	4	5	7	8	10	11	12
·78	6026	6039	6053	6067	6081	6095	6109	6124	6138	6152	1	3	4	6	7	8	10	11	13
·79	6166	6180	6194	6209	6223	6237	6252	6266	6281	6295	1	3	4	6	7	9	10	11	13
·80	6310	6324	6339	6353	6368	6383	6397	6412	6427	6442	1	3	4	6	7	9	10	12	13
·81	6457	6471	6486	6501	6516	6531	6846	6561	6577	6592	2	3	5	6	8	9	11	12	14
·82	6607	6622	6637	6653	6668	6683	6699	6714	6730	6745	2	3	5	6	8	9	11	12	14
·83	6761	6776	6792	6808	6823	6839	6855	6871	6887	6902	2	3	5	6	8	9	11	13	14
·84	6918	6934	6950	6966	6982	6998	7015	7031	7047	7063	2	3	5	6	8	10	11	13	15
·85	7079	7096	7112	7129	7145	7161	7178	7194	7211	7228	2	3	5	7	8	10	12	13	15
·86	7244	7261	7278	7295	7311	7328	7345	7362	7379	7396	2	3	5	7	8	10	12	13	15
·87	7413	7430	7447	7464	7482	7499	7516	7534	7551	7568	2	3	5	7	9	10	12	14	16
·88	7586	7603	7621	7638	7656	7674	7691	7709	7727	7745	2	4	5	7	9	11	12	14	16
·89	7762	7780	7798	7816	7834	7852	7870	7889	7907	7925	2	4	5	7	9	11	13	14	16
·90	7943	7962	7980	7998	8017	8035	8054	8072	8091	8110	2	4	6	7	9	11	13	15	17
·91	8128	8147	8166	8185	8204	8222	8241	8260	8279	8299	2	4	6	8	9	11	13	15	17
·92	8318	8337	8356	8375	8395	8414	8433	8453	8472	8492	2	4	6	8	10	12	14	15	17
·93	8511	8531	8551	8570	8590	8610	8630	8650	8670	8690	2	4	6	8	10	12	14	16	18
·94	8710	8730	8750	8770	8790	8810	8831	8851	8872	8892	2	4	6	8	10	12	14	16	18
·95	8913	8933	8954	8974	8995	9016	9036	9057	9078	9099	2	4	6	8	10	12	15	17	19
·96	9120	9141	9162	9183	9204	9226	9247	9268	9290	9311	2	4	6	8	11	13	15	17	19
·97	9333	9354	9376	9397	9419	9441	9462	9484	9506	9528	2	4	7	9	11	13	15	17	20
·98	9550	9572	9594	9616	9638	9661	9683	9705	9727	9750	2	4	7	9	11	13	16	18	20
·99	9772	9795	9817	9840	9863	9886	9908	9931	9954	9977	2	5	7	9	11	14	16	18	20

TABLE VI.

Angle.	Sine.	Cosine.	Tangent.	Cotangent.	Radians.		Angle.
0°	0	1	0	∞	0	1·5708	**90°**
1	·0175	·9998	·0175	57·2900	·0175	1·5533	89
2	·0349	·9994	·0349	28·6363	·0349	1·5359	88
3	·0523	·9986	·0524	19·0811	·0524	1·5184	87
4	·0698	·9976	·0699	14·3006	·0698	1·5010	86
5	·0872	·9962	·0875	11·4301	·0873	1·4835	**85**
6	·1045	·9945	·1051	9·5144	·1047	1·4661	84
7	·1219	·9925	·1228	8·1443	·1222	1·4486	83
8	·1392	·9903	·1405	7·1154	·1396	1·4312	82
9	·1564	·9877	·1584	6·3138	·1571	1·4137	81
10	·1736	·9848	·1763	5·6713	·1745	1·3963	**80**
11	·1908	·9816	·1944	5·1446	·1920	1·3788	79
12	·2079	·9781	·2126	4·7046	·2094	1·3614	78
13	·2250	·9744	·2309	4·3315	·2269	1·3439	77
14	·2419	·9703	·2493	4·0108	·2443	1·3265	76
15	·2588	·9659	·2679	3·7321	·2618	1·3090	**75**
16	·2756	·9613	·2867	3·4874	·2793	1·2915	74
17	·2924	·9563	·3057	3·2709	·2967	1·2741	73
18	·3090	·9511	·3249	3·0777	·3142	1·2566	72
19	·3256	·9455	·3443	2·9042	·3316	1·2392	71
20	·3420	·9397	·3640	2·7475	·3491	1·2217	**70**
21	·3584	·9336	·3839	2·6051	·3665	1·2043	69
22	·3746	·9272	·4040	2·4751	·3840	1·1868	68
23	·3907	·9205	·4245	2·3559	·4014	1·1694	67
24	·4067	·9135	·4452	2·2460	·4189	1·1519	66
25	·4226	·9063	·4663	2·1445	·4363	1·1345	**65**
26	·4384	·8988	·4877	2·0503	·4538	1·1170	64
27	·4540	·8910	·5095	1·9626	·4712	1·0996	63
28	·4695	·8829	·5317	1·8807	·4887	1·0821	62
29	·4848	·8746	·5543	1·8040	·5061	1·0647	61
30	·5000	·8660	·5774	1·7321	·5236	1·0472	**60**
31	·5150	·8572	·6009	1·6643	·5411	1·0297	59
32	·5299	·8480	·6249	1·6003	·5585	1·0123	58
33	·5446	·8387	·6494	1·5399	·5760	·9948	57
34	·5592	·8290	·6745	1·4826	·5934	·9774	56
35	·5736	·8192	·7002	1·4281	·6109	·9599	**55**
36	·5878	·8090	·7265	1·3764	·6283	·9425	54
37	·6018	·7986	·7536	1·3270	·6458	·9250	53
38	·6157	·7880	·7813	1·2799	·6632	·9076	52
39	·6293	·7771	·8098	1·2349	·6807	·8901	51
40	·6428	·7660	·8391	1·1918	·6981	·8727	**50**
41	·6561	·7547	·8693	1·1504	·7156	·8552	49
42	·6691	·7431	·9004	1·1106	·7330	·8378	48
43	·6820	·7314	·9325	1·0724	·7505	·8203	47
44	·6947	·7193	·9657	1·0355	·7679	·8029	46
45°	·7071	·7071	1·000	1·000	·7854	·7854	**45°**
Angle.	Cosine.	Sine.	Cotangent.	Tangent.	Radians.		Angle.

MAY EXAMINATION OF THE DEPARTMENT OF SCIENCE AND ART, 1899.

ELEMENTARY PRACTICAL MATHEMATICS.

EXAMINER: PROFESSOR JOHN PERRY, D.SC., F.R.S.

INSTRUCTIONS.

If the rules are not attended to, the paper will be cancelled.

Put the number of the question before your answer.

You are to confine your answers *strictly* to the questions proposed.

Your name is not given to the Examiner, and you are forbidden to write to him about your answers.

You may not answer more than SEVEN questions.

A table of logarithms and functions of angles and useful constants is supplied to each candidate.

The examination in this subject lasts for three hours.

1. Using either a slide rule or table of logarithms, or by contracted arithmetical methods, calculate $23{\cdot}51 \times 6{\cdot}78$, $23{\cdot}51 \div 6{\cdot}78$, $23{\cdot}51 \div {\cdot}0678$, $\sqrt[3]{23{\cdot}51}$, $6{\cdot}78^{2{\cdot}34}$, $0{\cdot}678^{-1{\cdot}301}$.

Why is it that we add logarithms of numbers to obtain the logarithm of their product? Why do we divide the logarithm of a number by 2 to obtain the logarithm of its square root?

2. Answer *only one* of the following questions, (*a*), (*b*), or (*c*):

(*a*) In the following formula, $a=25$, $b=8{\cdot}432$, $c=0{\cdot}345$, $\theta=0{\cdot}4226$ radians; find the value of

$$a^{1{\cdot}157}b^{-\frac{1}{3}} \div \theta(c^3 + a\log_e b \,.\, \tan\theta).$$

(*b*) Work out the values of

$$(sr^{-1} - r^{-s}) \div (s-1) \text{ and } (1 + \log_e r) \div r$$

when $s=0{\cdot}95$, and $r=1{\cdot}75$.

(*c*) If $pu^{1{\cdot}0646}=479$, find p when u is $12{\cdot}12$; find u when p is $60{\cdot}4$.

3. Answer *only one* of the following questions, (*a*), (*b*), or (*c*):

(*a*) A cubic cm. of mercury weighs 13·6 grammes; obtain the equivalent of a pressure of 760 mm. of mercury in inches of mercury, in feet of water, in pounds per square inch and per square foot, and in kilogrammes per square cm.

(*b*) A right circular cone was measured. The method of measurement was such that we only know that the diameter of base is not less than 6·22 nor more than 6·24 inches, and the slant side is not less than 9·42 nor more than 9·44 inches. Find the slant area of the cone taking (1) the lesser dimensions, (2) the greater dimensions. Express half the difference of the two answers as a percentage of the mean of the two.

In calculating the area, if a man gives 10 significant figures in his answer, how many of these are unnecessary?

(*c*) 260 feet of round copper wire weighs 3 lb.; find its diameter if a cubic inch of the copper weighs 0·32 lb. If the same weight of the copper is shaped like a hollow cylinder, 1 inch internal diameter and 2 inches long, what is its external diameter?

4. Answer *only one* of the following, (*a*) or (*b*):

(*a*) The tangent of an angle is 0·675; draw the angle, without using tables, and explain your construction.

Along the lines forming the angle set off lengths $OA = 4{\cdot}23''$ and $OB = 3{\cdot}76''$. Find the length AB, either by measurement or by calculation.

(*b*) The lengths of two sides of a triangle are 3·8 and 4·6 inches, and the angle between them is 35°; determine by drawing or in any way you please (1) the length of the third side, and (2) the area of the triangle.

5. Answer *only one* of the following, (*a*), (*b*), or (*c*):

(*a*) By using squared paper or by any other method, divide the number 420 into two parts such that their product is a maximum. Describe your method.

(*b*) If H is proportional to $D^{\frac{2}{3}}v^3$, and if H is 871 when D is 1330 and v is 12, find H when D is 1200 and o is 15.

(*c*) Calculate the ratio of d_0 to d_1 from the equation

$$d_0^3 = 1{\cdot}3\,\frac{d_0^4 - d_1^4}{d_0}.$$

6. Answer *only one* of the following questions, (*a*) or (*b*):

(*a*) A body like the trunk of a tree, 13 feet long, its axis being straight, has the following cross sectional areas of A square inches at the following distances, x inches from its end. Find its volume, using squared paper.

The following table gives the value of A for each value of x:

x	0	20	35	56	72	95	110	140	156
A	405	380	362	340	325	304	287	260	252

(*b*) The transverse sections of a vessel are 15 feet apart and their areas in square feet up to the load water line are 4·8, 39·4, 105·4, 159·1, 183·5, 173·3, 127·4, 57·2, and 6·0 respectively. Find the volume of water displaced by the ship between the two end sections given above.

7. Find the distance of the centre of gravity, from the end from which x is measured in (*a*) Question 6.

8. Answer *only one* of the following, (*a*) or (*b*):

(*a*) The following values of two quantities, which we may call x and y, were measured. Thus when x was found to be 1, y was found to be ·223.

x	1	1·8	2·8	3·9	5·1	6·0
y	·223	·327	·525	·730	·910	1·095

It is known that there is a law like

$$y = a + bx$$

connecting these quantities, but the observed values are slightly wrong. Plot the values of x and y on squared paper and find the most likely values of a and b.

(*b*) The following measurements were made at an Electric Light Station under steady conditions of output:

W is the weight in pounds of feed water per hour, and P the electric power, in kilowatts, given out by the station. When P was 50, W was found to be 3800; and when P was 100, W was found to be 5100.

If it is known that the following law is nearly true

$$W = a + bP,$$

using squared paper or any way you please, find W when P is 70 kilowatts.

State the value of $W \div P$ in each of the three cases.

9. The population P (millions) of a certain country is ascertained at the beginning of each of these years to be

Year	1830	1835	1840	1850	1860	1865	1870	1880	1890	1895
P	20	22·1	23·5	29·0	34·2	38·2	41·0	49·4	57·7	63·3

Let t be the time in years from 1830. Show the relation of P and t by a simple curve going fairly evenly through the plotted points. Find what the population probably was at the beginning of 1848, and the rate of increase of population per annum then.

10. Answer *only one* of the following questions, (*a*), (*b*), or (*c*):

(*a*) Define curvature and radius of curvature.

The length of a railway curve which has a uniform curvature is one mile, and the total change of direction is 30 degrees. Find the value of the curvature and of the radius of curvature.

(*b*) Define *radian* and *angular velocity*.

A wheel makes 90 turns per minute; what is its angular velocity in radians per second? If a point on the wheel is 6 feet from the axis, what is its linear speed?

(*c*) The earth being supposed spherical, of 4000 miles radius, what is the linear velocity in miles per hour of a point in 36° North latitude? The earth makes one revolution in 23·93 hours.

11. Describe any system which you know of that enables us to define exactly the position of a point in space.

Work *one* but not *both* of the following, (*a*) or (*b*):

(*a*) The three rectangular coordinates of a point P are 3, 4, and 5; determine (1) the length of the line joining P to O, the origin of coordinates; (2) the cosines of the angles which OP makes with the three rectangular axes.

(*b*) The polar coordinates of a point are

$$r=3, \quad \theta=65^\circ, \quad \phi=50^\circ.$$

Determine its rectangular coordinates.

12. Only *one* of the following, (*a*) or (*b*), is to be answered:

(*a*) A straight line 4 inches long makes 30° with the horizontal plane of projection, and 50° with the vertical plane. Draw its projections on these planes; measure and write down their lengths.

(*b*) Represent by its traces a plane inclined 78° and 35° respectively to the horizontal and vertical planes of projection. Measure and write down the angles which the traces make with the line in which the planes of projection intersect each other.

13. There are three vectors in one plane:

A of the amount 2 in the direction towards the North-East.

B of the amount 3 in the direction towards the North.

C of the amount 2·5 in the direction towards 20° East of South.

By drawing, or any methods of calculation, find the following vector sums and differences:

$$A+B+C, \quad B+C-A, \quad A-C.$$

14. Answer *only one* of the following questions, (*a*) or (*b*):

(*a*) What is meant by the slope of a curve *at a point* on the curve? How is this measured? If the coordinates of points on the curve represent two varying quantities, say, distance and time, what does the slope of the curve at any place represent? Prove your statement.

(*b*) At the end of a time t it is observed that a body has passed over a distance s reckoned from some starting-point. If it is known that $$s=20+12t+7t^2,$$

find s when t is 5, and by taking a slightly greater value of t, say 5·001, calculate the new value of s and find the average velocity during the ·001 second. How would you proceed to find the exact velocity at the instant when t is 5, and how much is this velocity?

15. Plot *only one* of the following curves from $x=0$ to $x=8$. To calculate five values of y will probably be enough:

$$y=4x^{0\cdot70}, \quad\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$y=2\cdot3\sin\left(\cdot2618x+\frac{\pi}{6}\right), \quad\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

$$y=0\cdot53e^{0\cdot26x}. \quad\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

Also give the answer to *only one* of the following questions, (*a*) or (*b*), using any method of calculation you please:

(*a*) In the case chosen by you find the rate of increase of y with regard to x where $x=3$; or

(*b*) find the average value of y from $x=0$ to $x=8$.

16. Find a value of x which satisfies any one of the following equations you please. *Only one* is to be attempted:

$$2x^{3\cdot1}-3x-16=0.$$

$$2\cdot42x^3-3\cdot15\log_e x-20\cdot5=0.$$

$$e^x-e^{-x}+0\cdot4x-10=0.$$

The answer to be given correctly to *three* significant figures.

ANSWERS.

Exercise I., p. 19.

1. 3 to 2. **2.** 1 : ·417. **3.** 11200 cub. in. **4.** 9800.
5. ·9. **6.** 75 to 1024; 81 to 16. **7.** ·03288. **8.** 1·175.
9. ·85. **10.** 7·7. **11.** ·298. **12.** 323130 grms.
14. ·8. **15.** 2·16. **16.** ·946. **17.** 1·26. **18.** 450.

Exercise II., p. 32.

1. 1·5708; 90°; ·65014. **2.** 3·14159; ·7854; ·4392.
3. 3·1416; ·3708; 20° 14′ 42″. **4.** 19°·098. **5.** 18·85.
6. 3·77 ins. **7.** 2·094. **8.** 137·5 ins. **9.** 1·152 miles.
10. 14; 600 ft. **12.** 12·36. **13.** 1·3963; 51° 25′ 48″. **14.** 70°·01.
15. 66°. **16.** 132°. **17.** 5°·7. **18.** 120°; 2·094. **19.** 22°·4; ·3909.
20. 1; 38·23. **21.** 17·19 ft. **22.** (i) ·63274; (ii) 137·51; (iii) 2·909.

Exercise III., p. 34.

1. 90°; 36° 48′; 53° 12′. **2.** A = 31° 53′; B = 8° 10′; C = 139° 56′.
3. 55° 46′. **4.** 142; 58° 32′. **5.** 240·2; 39° 15′.
6. 377; 381·5. **7.** 230·9; 116°; 27°. **8.** 42° 48′; 51°; 86° 12′.

Exercise IV., p. 42.

1. 177. **2.** 1507. **3.** 144. **4.** 36. **5.** 47. **6.** 13.
7. 11; 67. **8.** 37. **9.** 87. **10.** 37. **11.** 493. **12.** 43.
13. 125. **14.** 677. **15.** 379. **16.** 881. **17.** 1849.
18. 571. **19.** 77. **20.** 2571. **21.** 1350. **22.** 72.
23. 12. **24.** 11. **25.** 26. **26.** 74. **27.** 20493.

Exercise V., p. 44.

1. 120. **2.** 528. **3.** 13728. **4.** 3264. **5.** 2970.
6. 1729728. **7.** 67. **8.** 19152. **9.** 7560. **10.** 22176.
11. 1260. **12.** 2640. **13.** 63072. **14.** 2520. **15.** 2788.

16. 52948. **17.** 1008. **18.** 15120. **19.** 780. **20.** 12600.
21. 4500. **22.** 39270. **23.** 306. **24.** 35280. **25.** 7956.
26. 1111880. **27.** 432432. **28.** 202125. **29.** 888.

Exercise VI., p. 47.

1. $\frac{9}{19}$, $\frac{7}{15}$, $\frac{2}{5}$, $\frac{3}{10}$; $\frac{33}{190}$. **2.** $\frac{7}{10}$, $\frac{11}{16}$, $\frac{5}{8}$, $\frac{4}{7}$; $\frac{9}{70}$. **3.** 1.
4. 1. **5.** $25\frac{7}{52}$. **6.** $1\frac{2}{25}$. **7.** $5\frac{659}{720}$. **8.** $7\frac{221}{224}$. **9.** $12\frac{9}{16}$.
10. $1\frac{13}{14}$. **11.** £7173. 6s. 8d., £8070, £8608, £8966. 13s. 4d.
12. $\frac{15}{16}$. **13.** $3\frac{31}{120}$. **14.** $\frac{80}{81}$, $\frac{11}{12}$, $\frac{4}{27}$, $\frac{1}{9}$. **15.** 12 minutes.
16. $\frac{1}{60}$. **17.** $1\frac{1}{11}$ hours. **18.** $8\frac{23}{42}$. **19.** £58. 1s. 4d.

Exercise VII., p. 50.

1. $\frac{9}{40}$. **2.** $\frac{21}{55}$. **3.** 1. **4.** $3\frac{1}{2}$. **5.** $\frac{65}{333}$. **6.** $\frac{11}{78}$.
7. 2. **8.** 8. **9.** 1. **10.** 11400000, 17100000, 3800000.
11. 221. **12.** £4. 4s. 11d. **13.** £2. 7s. 6d.
14. £1. 2s. 6d.; £1. 16s.; £1. 4s. **15.** £1813. 6s. 8d. **16.** £30750.

Exercise VIII., p. 52.

1. 169·2095. **2.** 3·0605. **3.** ·0332. **4.** ·0086.
5. 1·8806. **6.** 714·1791. **7.** 984·5643. **8.** 2873·1007.
9. 241·7614. **10.** ·2531. **11.** 756·8719. **12.** 5208·7881.
13. 89·0122. **14.** ·9148. **15.** ·0468. **16.** 210·9551.

Exercise IX., p. 54.

1. ·00161. **2.** 82·074. **3.** 32046. **4.** ·4007.
5. 4632700. **6.** ·0651. **7.** 34·67. **8.** ·078.
9. ·005. **10.** ·0435. **11.** 15000. **12.** ·0003.
13. 2460. **14.** 3810·607. **15.** 13·75. **16.** ·00376.

Exercise X., p. 56.

1. 5·2192. **2.** 26·64. **3.** 4124·952. **4.** 1·032.
5. 5631·585. **6.** 76917·06. **7.** 224·54. **8.** 1048465·32.
9. 469·4514. **10.** 24·36. **11.** 3·8. **12.** 30·8.

Exercise XI., p. 56.

1. 6·538. **2.** 13·26. **3.** ·0046. **4.** ·0309. **5.** 13·279.
6. ·0209. **7.** ·0481. **8.** 2. **9.** 3·773. **10.** 80.
11. ·9941 metre. **12.** (i) $9\frac{3}{7}$, (ii) $\frac{123}{295}$. **13.** (i) ·3391, (ii) ·8125.

Exercise XII., p. 57.

1. 11 st. $13\frac{1}{16}$ lbs. **2.** 71·8. **3.** ·293. **4.** £·79, £·44. **5.** 1580.

Exercise XIII., p. 62.

1. 1·558 ; ·04042. 2. ·4304712. 3. 38·851. 4. 48·01.
5. 1826·3919. 6. 204·076. 7. ·1883. 8. 47975·1.
9. ·00752743. 10. ·008739. 11. ·0225512. 12. ·00334931.

Exercise XIV., p. 63.

1. 625. 2. 54289. 3. 31·855.
4. 121·287. 5. 1369, 50653. 6. 94·09, 912·673.

Exercise XV., p. 66.

2. 14·07. 3. 7963. 4. ·1418. 5. 2·5 or $2{\cdot}4\dot{9}$.
6. 40795. 7. 82929. 8. 503. 9. 671·3.
10. 7963. 11. $2{\cdot}09\dot{3}$. 12. $1\frac{1}{6}$. 13. 7·0711. 14. ·1123.

Exercise XVI., p. 67.

1. 1023·8 feet. 2. 3190 inches. 3. ·17029.
4. 171·61 sq. feet. 5. ·7142, ·2258. 6. 17·8885. 7. 2·887, ·379.

Exercise XVII., p. 73.

1. £2. 6s. 3d. 2. £62. 10s. 3. £875. 4. 200 : 243.
5. $4\frac{1}{8}$. 6. £446. 10s. 7. 2s. 10·9d. 8. 2825761, 2560000.
9. 25 per cent. 10. 10s. $11\frac{1}{4}$d. 11. $18\frac{2}{9}$. 12. $34\frac{4}{9}$ per cent.

Exercise XVIII., p. 79.

1. $6\frac{1}{4}$. 2. 0. 3. 7. 4. 1. 5. 9. 6. 12.
7. 19. 8. 27. 9. 64. 10. 5. 11. 48. 12. 64.
13. 13. 14. $1\frac{1}{3}$. 15. $\frac{24}{115}$. 16. (*a*) 4, (*b*) $2\frac{2}{5}$, (*c*) $2\frac{5}{8}$.
17. (*a*) $-29\frac{2}{3}$, (*b*) $-3\frac{1}{12}$, (*c*) $3\frac{19}{77}$. 18. (*a*) $\frac{11}{168}$, (*b*) $60\frac{23}{27}$, (*c*) $-\frac{18}{65}$.

Exercise XIX., p. 81.

1. $6x$. 2. $10x-4y+1$. 3. $\frac{-x+9y-3}{2}$. 4. $16a^4x^4-a^8$.
5. $2x+4b-6c$. 6. 1. 7. $x^2-(a+b)x+ab$. 8. 94.
10. $4a^2-9b^2+24bc-16c^2$. 11, $2(a-3b)$. 12. $\frac{11a}{5}-2b$; 59.

Exercise XX., p. 83.

1. $2a+6c$. 2. $11x^2-9x+9$. 3. $3xy-2ab$.
4. x^3-x^2-2c. 5. $6xy+5az-bc+5mn-3p$.
6. $9ab-4yz-7cx+4ek-8$. 7. $5b+2c$; $4a-7c$.
8. $4ab-a^2-5b^2$. 9. $a^3+3b^3+ab^2$.

Exercise XXI., p. 83.

1. $3a-5b+2c$. **2.** $3z+5y-y^2-2x+30$.
3. $3x^3-3x^2+3x-3$; 60. **4.** $xy(b-c)$.
5. $10mz+4bx-10ay+1$. **6.** (*a*) $x^3-6x^2y+11xy^2-10y^3+4y^2$; (*b*) a.
7. $a+10b-13c+5d$. **8.** $7a-7b$. **9.** $3x^3-3x^2+3x-3$.
10. $xy(b-c)$. **11.** $20x-21y-9$. **12.** $4x^2(7x-13)$. **13.** 4.

Exercise XXII., p. 86.

1. $x^5+4x^4+48x-32$. **2.** $36x^6-217x^4+416x^2-225$.
3. $4a^2-16ac+16c^2-9b^2$. **4.** x^6-1.
5. $16x^8+36x^6y^2+81x^4y^4$. **6.** $x^3+y^3+z^3-3xyz$.
7. $\frac{x^3}{y^3}-\frac{y^3}{x^3}$. **8.** $x^4+\frac{625}{4}$.
9. $36x^6-217x^4+406x^2-225$. **10.** x^6-1.
11. $16x^8+36x^6y^2+81x^4y^4$. **12.** $\frac{x^3}{2}+\frac{y^3}{2}+\frac{z^3}{2}-\frac{3}{2}xyz$.
13. x^6-16x^3+64. **14.** $a^5-7a^3x^2+13a^2x^3-x^5$.

Exercise XXIII., p. 87.

1. $7x^2+4x-5$. **2.** x^2-3x-y. **3.** $3x^2-x+2$.
4. $3a^2-ax-2x^2$. **5.** $8x^3+12x^2y+6xy^2+y^3$.
6. $a^3-3a^2x+3ax^2-x^3$. **7.** $x^2+2ax-bx+a^2-ab+b^2$.
8. $7x^2+4x-5$. **9.** x^2-3x-y.
10. $8x^3+12x^2y+6xy^2+y^3$. **11.** $\frac{7a-12b+6c}{4}$.
12. $x^2+2ax-bx+a^2-ab+b^2$. **13.** $x-1+\frac{1}{x}$.
14. (i) $x^3-3x^2+11x-16$; (ii) $x^2+y^2+z^2-xy-xz-yz$.
15. $3x^2-7x+9$.

Exercise XXIV., p. 90.

1. (i) $(x+3y)(x^2-3xy+9y^2)$; (ii) $(x-5)(x-8)$.
2. $(a-x)(a+x)(a^2+x^2)(a^4+x^4)$. **3.** $x^3 . x+1$.
4. (i) $(3x-8)(3x+8)$. (ii) $(x+1)(x^2-x+1)$. (iii) $x(x+2)(x^2+2x+2)$.
5. $\frac{2x^2+3x-1}{x^2-3x-2}$. **6.** (i) $\frac{34xy}{49x^2-y^2}$. (ii) $\frac{a^2b-a^2c+b^2c+c^2a-ab^2-bc^2}{abc(a-b)(b-c)(c-a)}$.
7. (i) $(x+13)(x-6)$. (ii) $(x-9)(y+4)$.
(iii) $(x+1)(x-1)(y+1)(y-1)$. (iv) $(a-b+c-d)(a+b-c-d)$.
8. $(x+3)(x-3)(x+2)$. **9.** $(a+b)(2a+b)(3a+b)$.
10. (i) $(2x-3)(x+4)$. (ii) $(x+y+z)(x+y-z)$. (iii) $(x^2-y^2)(x+y)$.
11. $x^6-14x^4+49x^2-36$; $7x^2-2xy+y^2$. **12.** 1.
13. (i) $(x+2)(x-1)$. (ii) $(x-3y+z)(x-3y-z)$.
(iii) $(x+a)(x-a)(2x+3a)$.

14. (i) $(x+10)(x-3)$. (ii) $10x(x+2y)(x-2y)$. (iii) $(x^2+y^2)(x+y)$.
15. (i) $(x-2)(x+1)$. (ii) $(x^2-2xy+4y^2)(x^2+2xy+4y^2)$.
(iii) $(x-y)(x+y+6)$.
16. (i) $\left(\frac{a}{4}-3b\right)^2$. (ii) $(x+y-1)(x+y-2)$. (iii) $(x+y+1)(x-y+4)$.

Exercise XXV., p. 93.

1. $8x^3-12x^2y+18xy^2-27y^3$. **2.** $a^{4(p+q)}$. **3.** $a^{12(m-n)}$.
4. $\frac{a^p}{3b^q}$. **5.** $\sqrt[13]{(x^{38}y^{24}z^{10})}$. **6.** $b^{\frac{3}{2}}$.
7. $a^{3(p+q)}$. **8.** $a^{-\frac{3}{2}}b^{-\frac{7}{12}}$. **9.** $a^{9(n-m)}$.

Exercise XXVI., p. 93.

1. (i) $x^6-3x^5+9x^4-27x^3+81x^2-243$; (ii) $1+5x+10x^2+20x^3+40x^4$.
3. $1+2x+3x^2+4x^3$. **4.** $3x^3-2x^2-5x-3$.
5. $3x^2-4y+7$. **6.** $x^2-2xy+3xz+6yz$.
7. $x^4+2ax^3+3a^2x^2+2a^3x+8a^4$; $4x^2-3ax+a^2$.
8. $6x^8-11x^6+22x^4-4x^2-7$; $3x^2-2x-9$.
9. (i) $(x+4)(x-18)$; (ii) $(x-1)(x+3a)$; (iii) $(x-3)(x^2+3x+9)$.
10. (i) $\frac{x}{x-3}$; (ii) $\frac{x-2}{(x+1)(x^2-x+1)}$.
11. $x^3-9x^2y+26xy^2-30y^3$; $5x^4+4x^3+3x^2+2x$.
12. (i) $(5x-2)(2x+1)$; (ii) $(x-17)(x+15)$; (iii) $\left(x-\frac{3}{2}a\right)(6x+4b)$.
13. (i) $\frac{a^2}{a^2-x^2}$; (ii) x^2. **14.** $\frac{x(yz+1)+z}{x(yz+1)-z}$.
15. $a-(b-c)$; $a-b+c$. **16.** (i) 1; (ii) $26\frac{3}{4}$.
17. (i) $2xy(x^2y^2-1)$; (ii) $\frac{x^2}{4}-6xy+\frac{y^2}{9}$.
18. (i) $\frac{1}{2x(3+x^2)}$; (ii) $\frac{2a}{3}$. **19.** (i) $\frac{1}{2x^2-1}$; (ii) a; (iii) $2a^2$.

Exercise XXVII., p. 97.

1. 1·0001; ·992. **2.** ·997. **3.** 1·003. **4.** ·9967. **5.** 1·008.
6. ·998. **7.** ·9975. **8.** 1·04. **9.** 1·04; 1·02. **10.** ·96; ·98.
11. ·96; 1·04. **12.** ·98; 1·02. **13.** 1·03; ·97; ·97; 1·03.

Exercise XXVIII., p. 100.

1. x^2-x-1. **2.** $x+6$. **3.** $x-1$. **4.** $2x^2-3$.
5. $2x^3-4x^2+x-1$. **6.** $x+2$. **7.** $2x^2-x-3$. **8.** $3(x^2-x-2)$.
9. x^2-3x+1. **10.** $x+3$. **11.** x^2+1. **12.** $x-9$.
13. $x-1$. **14.** $2x-3$. **15.** x^2+2x-3. **16.** $x-1$.
17. $x+y$. **18.** $x+4$. **19.** $3x-2$. **20.** $2a-3x$.

Exercise XXIX., p. 102.

1. $(a^2-x^2)(a^2-4x^2)$. **2.** $a^4b^3c^3(x-y)(x-3y)(3x-y)$.
3. $x^4-3a^2x^2-4a^4$. **5.** $288x^3y^3(x-y)(x^3-y^3)$.
6. $2x^6+2x^5+17x^4+14x^3+53x^2+32x+48$. **7.** $(x-1)(x-2)(x-3)$.
8. $180a^3(a+x)^3(a-x)^3$. **9.** x^6-y^6. **10.** $30a^4b^3x^3y^4$.
11. $600a^5x^5(a^3-x^3)(a+x)$. **12.** $240x^4y^2(x+y)^2(x-y)$.
13. $60x^3y^4(x-y)^3(x+y)(x^2+xy+y^2)$. **14.** $3888x^3y(x^3-y^3)(x+y)$.
15. $1188xy^7(x-y)(x-2y)(x-3y)$.

Exercise XXX., p. 104.

1. $\frac{(x+1)^2}{(x-2)^2}$. **2.** $\frac{x}{9}$. **3.** $\frac{abc}{xyz}$. **4.** x^2+a^2. **5.** $2x$.
6. 1. **7.** $\frac{4x-3a}{4a^2}$. **8.** $\frac{2ax^2}{3}$. **9.** 1. **10.** $4(a+x)$.
11. $\frac{3}{y}$. **12.** $5x^3$. **13.** $\frac{a}{2(a+y)}$. **14.** xy. **15.** $2ab^2$.
16. $\frac{ab+a-b-1}{ab-2a-2b+4}$. **17.** $\frac{9+x^2}{2-2x^2}=7$. **18.** $-\frac{1}{2a-1}$. **19.** $\frac{x+y}{1-xy}$.
20. $\frac{x^3}{1+x^3}$. **21.** $-\frac{1}{x}$. **22.** $\frac{a^4+b^4}{a^4-b^4}$. **23.** $3y-2x$. **24.** $\frac{a(2a+6)}{(2a-b)^2}$.
25. $(x-1)^2$. **26.** $\frac{x^2+y^2}{2(x^4+x^2y^2+y^4)}$. **27.** $\frac{xy}{x+y}$. **28.** $\frac{x-2}{x-1}$.

Exercise XXXI., p. 107.

1. $5x^4-6x^2-7$. **2.** $x^2+4x-21$. **3.** a^2-a+4. **4.** $4x^2-5x+8$.
5. $5x^2-3x+4$. **6.** $2a^2-3ax+x^2$. **7.** $2x^2-x+\frac{1}{2}$. **8.** x^2+3x-2.
9. $\frac{y}{x}+1+\frac{x}{y}$. **10.** $\frac{2x^2}{3}-6x+1$. **11.** $x^2+1+\frac{1}{x^2}$.
12. $\frac{2x}{y}-\frac{2y}{x}$. **13.** $3x^2-5xy+y^2$.

Exercise XXXII., p. 107.

1. $3x^2-2x-7$. **2.** x^2+3; $(x^2+3)(4x-9)$; $(x^2+3)(2x^2-3x-2)$.
3. $(2x^2-x+1)$; $(x-1)(2x+1)(2x^2-x+1)$; $(x+3)(2x^2-x+1)$.
4. $2x^2+1$. **5.** x^2-5x+1; $(x^2-5x+1)(x^2-7)$; $(x^2-5x+1)(3x-8)$.
6. $2x^2+2x+3$. **7.** $3y(3x+2y)$. **8.** $3x-7$.
9. $x-a$; $\frac{3x-a+2b}{(x-a)(x-b)}$. **10.** $\frac{x+y}{y}$. **11.** $\frac{x^4-x^2y^2+y^4}{x^2-y^2}$.
12. $\frac{2xy}{x^2-y^2}$. **13.** $\frac{y}{y^2-9x^2}$. **14.** $x-y$. **15.** $\frac{2}{a+2}$.
16. $\frac{x-1}{x+1}$. **17.** $m+n-a$. **18.** $9x-\frac{3y}{2}$.

19. $\frac{8xy}{x^2-4y^2}$; $-\frac{1}{ab}$; $-\frac{8}{3}$. **21.** $\frac{4x+3}{3x^2-5x+7}$; $\frac{1}{35}$.

22. (i) $(2x-1)(x+2)$; (ii) $12(x-2)(2x-1)$; (iii) $(x-2)(x+2)(2x-1)$; L.C.M. $=12(x-2)(x+2)(2x-1)$.

23. L.C.M. $=(x-2)(x+2)(x-6)(3x-2)$.

24. $36(x^2-1)(x^2-4)(x^2-9)$. **25.** $30(x-1)(x+3)(x-3)(x^2+x+1)$.

26. (i) $(5x+3y)(3x-5y)$; (ii) $(a+3b-c)(a-b+3c)$; (iii) $2x(3+x^2)$.

27. 0; c.

Exercise XXXIII., p. 114.

1. 30. **2.** $\frac{1}{3}$. **3.** 11. **4.** $\frac{2}{3}$. **5.** $\frac{3}{2}$.
6. 13. **7.** $3\frac{1}{2}$. **8.** $\frac{2}{3}$. **9.** 111. **10.** $\frac{1}{3}$.
11. $\frac{ab}{a+b}$. **12.** $\frac{3}{5}$. **13.** $\frac{2ab}{a+b}$. **14.** $a+3b$. **15.** $\frac{2a}{3}$.
16. $\frac{b-c}{2a}$. **17.** $\frac{a(2b-a)}{2a-b}$. **18.** 23. **19.** $\frac{a}{a+b}$. **20.** 4.
21. $5a=b$. **22.** 9·4. **23.** $\frac{m+n}{2}$. **24.** $\frac{2}{3}$.

Exercise XXXIV., p. 117.

1. 10. **2.** $3\frac{3}{4}$ days. **5.** 80, 40. **6.** 21, 3. **7.** 33, 18.
8. (a) $10\frac{10}{11}$ past 2; $43\frac{7}{11}$; $27\frac{3}{11}$. **9.** $c=1$.
10. 20 in.; 16 in. **11.** $23\frac{2}{5}$, $15\frac{3}{5}$. **12.** $4\frac{1}{2}$, $15\frac{3}{4}$ feet.
13. 100, 120. **14.** £7. 10s., £9, £7. 4s. **15.** 350.
16. 3, 14. **17.** £17. 6s. 8d., £25. 10s. **18.** 600; £500.

Exercise XXXV., p. 123.

1. $x=5$, $y=6$. **2.** $x=5$, $y=4$. **3.** $x=30$, $y=20$.
4. $x=3$, $y=\frac{1}{3}$. **5.** $x=\frac{3}{7}$, $y=\frac{7}{3}$. **6.** $x=\frac{3}{2}$, $y=\frac{2}{3}$.
7. $x=\frac{10}{5}$, $y=0$. **8.** $x=2$, $y=\frac{5}{2}$. **9.** $x=2$, $y=-1$.
10. $x=\frac{c(b+a)}{a+b}$, $y=\frac{c(b-a)}{a^2+b^2}$. **11.** $x=3$, $y=2$, $z=1$.
12. $x=5$, $y=6$, $z=7$. **13.** $x=12$, $y=12$, $z=12$.
14. $x=2$, $y=3$, $z=4$. **15.** $x=24$, $y=60$, $z=120$.
16. $x=6$, $y=10$, $z=9$. **17.** $x=7$, $y=0$. **18.** $x=5$, $y=-1$.
19. $x=1$, $y=3$. **20.** $x=3\frac{3}{10}$, $y=\frac{9}{20}$. **21.** $x=3$, $y=-2$.
22. $x=7{\cdot}1$, $y=2$. **23.** $x={\cdot}02$, $y=2{\cdot}9$. **24.** $x=2\frac{53}{107}$, $y=4\frac{256}{321}$.
25. $x=7\frac{1}{2}$, $y=6$. **26.** $x=11$, $y=8$. **27.** $x=5$, $y=5$.
28. $x=\frac{2}{3}$, $y=0$. **29.** $x=2$, $y=\frac{5}{2}$. **30.** 250. **31.** $x=\frac{3}{2}$, $y=\frac{2}{3}$.

Exercise XXXVI., p. 125.

1. $\frac{53}{112}$. **2.** 17 half crowns, 23 half sovereigns.
3. 5s. 6d. silk, 7s. 4d. satin per yard. **4.** 59. **5.** $\frac{29}{91}$.
6. $\frac{17}{21}$. **7.** (i) $a=17$, $b=-13$; (ii) 1·366. **8.** 1650.

Exercise XXXVII., p. 126.

1. (i) 25, (ii) $13\frac{40}{51}$. **2.** (i) 13, -11; (ii) $x=40$, $y=30$.
3. (i) 12, (ii) 7. **4.** *A*, £70; *B*, £35; *C*, £20.
5. 2s. 3d.; 3s. **6.** (i) 8, (ii) 1. **7.** $x=3$, $y=18$.
8. (i) $x=-\frac{1}{2}$, (ii) $1\frac{3}{4}$. **9.** (i) 3, (ii) 17. **10.** $x=4$, $y=6$.
11. (i) 19; (ii) $x=5\frac{1}{2}$, $y=7$. **12.** 5. **13.** 4·1 or ·7.
14. *A*, £4; *B*, £6; *C*, £480. **15.** $\frac{39}{47}$. **16.** 30000.
17. $a=17$, $b=31$. **18.** 25 feet; ·0245 inches.

Exercise XLI., p. 141.

1. 107. **2.** 45. **3.** 1 ft. $6\frac{1}{2}$ ins. **4.** 37 ft. **5.** 20.
6. 222·6. **7.** 327. **8.** 673·15 links. **9.** 39·6 ft.
10. 269·44 yds. **11.** 6 and 10 ft. **12.** 625·17. **13.** 850.

Exercise XLII., p. 149.

1. 1·289. **2.** ·004812. **3.** ·1476. **4.** 3·805. **5.** ·4346.
6. 38·88. **7.** 3·061. **8.** 8·826. **9.** 7·637. **10.** 133·8.
11. 1·113. **12.** ·05618. **13.** ·03106. **14.** ·6533.

Exercise XLIII., p. 150.

1. ·005146. **2.** ·03017. **3.** 82·08. **4.** 2230.
5. ·00874. **6.** 15000. **7.** ·0001601. **8.** 2272000.
9. 1850. **10.** 19·63. **11.** 469·8. **12.** ·00013.
13. 86400. **14.** 266·7. **15.** ·03118. **16.** 4250.
17. ·000001645; ·5099. **18.** ·00212; 21200.
20. 3·7304; 6·7304; $\bar{3}$·7304. **21.** ·8908; $\bar{2}$·0212.
22. $\bar{5}$·9092; 2·9358. **23.** 1·3457; 1·7531. **24.** 1·051; 9·8.
25. 6299; 643·7. **26.** $\bar{1}$·7962. **27.** 2·0258; $\bar{3}$·0258.

Exercise XLIV., p. 154.

1. $\bar{1}$·1505. **2.** $\overline{12}$·1072. **3.** 2·0258. **4.** $\bar{3}$·0258.
5. $\bar{2}$·9251. **6.** $\bar{1}$·2623. **7.** 1·8573. **8.** 1·1584.
9. $\bar{4}$·1761. **10.** 1·1999. **11.** ·1966; $\bar{4}$·8627; $\bar{2}$·5570.
12. ·6416; $\bar{1}$·0484; $\bar{5}$·6317. **13.** 3·4771; $\bar{4}$·5229; 3·3010; $\bar{4}$·3010.
14. $\bar{2}$·1072; $\bar{1}$·7015. **15.** 5·3979. **16.** ·9898; 6044.

17. ·3355 ; ·5 ; 1·5. **18.** 843·3. **19.** ·6386.
20. 1·413. **21.** ·4723. **22.** 1·302. **23.** ·4004.
24. 22·17. **25.** (i) 8·056 ; (ii) 5·893. **26.** 139·8.
27. 1·708 ; ·03634. **28.** 1·703. **29.** 20·94. **30.** 1599.
31. ·7288. **32.** ·5079. **33.** ·3532. **34.** 254·6.
35. ·4402. **36.** ·3348. **37.** (i) $\bar{1}$·0458 ; (ii) $\bar{3}$·0703 ; (iii) 12.
38. (i) 128700, 54·4 ; (ii) 10080, ·1224 ; (iii) ·5576, 2·283.
39. ·009170 ; 1·0788. **40.** $\frac{1}{8}$, $\frac{1}{2}$, 1, $\frac{3}{2}$.

Exercise XLV., p. 156.

1. ·4574 ; ·2505. **2.** ·00005082 ; ·001925.
3. 30040 ; ·08072 ; 39·16. **4.** 843·3. **5.** ·5544.
6. ·02443. **7.** 59·55. **8.** 47. **9.** ·3996.
11. ·5009. **13.** ·3960 ; ·0001171. **14.** $\overline{26}$·6890 ; $\bar{1}$·5809.
15. 1·3979 ; $\bar{2}$·4949 ; $\bar{1}$·6795. **16.** $\bar{1}$·7438.
17. ·1461 ; $\bar{2}$·1875 ; $\overline{10}$·1971. **18.** 4 ; $\bar{2}$·5378 ; 15·5.

Exercise XLVI., p. 168.

2. ·6981. **3.** 98° 12′ 48″ or 98°·213 ; 1·714.
4. ·3162 ; ·8943 ; 1 ; ·406 radians. **5.** $A=48°48'$; $B=59°$; $C=72°12'$.
6. (i) ·6246 ; (ii) $B=77°\ 7'\ 30''$; $C=28°\ 25'\ 30''$. **8.** $\frac{3}{5}$; $\frac{15}{4}$; $\frac{21}{5}$.

Exercise XLVII., p. 170.

1. 19° 28′ 16″. **5.** 18° ; ·3142 ; ·9511 ; ·3249. **7.** ·832 ; 1·803.
8. $\frac{\sqrt{5}-1}{4}$; $\frac{\sqrt{5}+1}{4}$. **9.** $\frac{1}{\sqrt{2}}$; $\sqrt{2}$. **10.** $\frac{3}{5}$; $\frac{4}{3}$; $-\frac{3}{5}$; $-\frac{4}{3}$. **11.** 7.
12. $\frac{4}{5}$; $\frac{5}{3}$; $\frac{24}{25}$; $-\frac{7}{25}$; $-\frac{24}{7}$. **14.** $\sin\theta=\frac{4}{5}$; $\tan\theta=\frac{4}{3}$.
15. (i) $\cos A=\frac{p^2-q^2}{p^2+q^2}$, $\operatorname{cosec} A=\frac{p^2+q^2}{2pq}$; (ii) 63° 6′.

Exercise XLVIII., p. 173.

1. ·5375. **2.** ·8100 ; 67° 30′ 30″ ; ·7903. **3.** $\bar{1}$·8495 ; ·6141.
4. ·4190. **5.** ·6362. **6.** ·2058 ; ·4429.
7. 7·5874. **8.** ·2479. **9.** 2·5 ; ·928 ; ·3714.

Exercise XLIX., p. 175.

1. 70·99. **3.** 57·7. **4.** 87·553. **5.** 392·92.
6. 107·24. **7.** 124·3. **8.** 106·65. **9.** 325·92.
10. 698·1. **11.** 40·363. **12.** 204·2. **13.** 117·7.
14. 448·5. **15.** 43·3. **16.** 120. **18.** 142·5. **19.** 189·54.
20. 236·55. **21.** 173·21. **22.** 137. **23.** 273·21. **25.** 732·1.

Exercise L., p. 191.

1. 62. **2.** 5. **3.** $25\frac{1}{5}$. **4.** – 62·5 ; – 120.
5. $-99\frac{3}{4}$. **6.** 630. **7.** 1, 3, 5 .... **8.** 9, 8, 7 .... **9.** $7\frac{1}{3}$.
10. 630. **11.** 25. **12.** $287\frac{1}{2}$. **13.** 156. **14.** 13·5.
15. 18. **16.** 80. **17.** 4864. **18.** $52\frac{1}{2}$. **19.** $52\frac{1}{2}$.
20. $22\frac{2}{3}$. **21.** $22\frac{2}{3}$. **22.** – 136·5. **23.** 25. **24.** 4890.

Exercise LI., p. 194.

1. 486 ; 728. **2.** 96 ; 189. **3.** 16, 24, 36, etc.
4. ·5395. **5.** – 15·86. **6.** 765.
7. – ·3035. **8.** $-99\frac{3}{4}$. **9.** 16, 24, 36, etc.
10. 2. **11.** 38·4. **12.** $\frac{3}{2}$. **13.** $\frac{16}{7}$. **14.** $\frac{4}{9}$. **15.** $\frac{28}{5}$. **16.** $\frac{7}{9}$.

Exercise LII., p. 195.

1. £321. 15s. 0d. **2.** £71. 10s. 0d. **3.** £4425. 0s. 0d.
4. 7·5 years. **5.** 9·25 years. **6.** £1478. 15s. 0d.
7. 10 years. **8.** £420. 11s. 0d. **9.** 8·25 years.
10. £2167. 10s. 0d. **11.** 5 %.

Exercise LIII., p. 197.

1. £249. 14s. 6·7d. **2.** £3. 10s. 0d. **3.** £118. 19s. $3\frac{1}{2}$d.
4. £441. 5s. 10·3d. **5.** £3. 15s. 0d. **6.** 12 yrs. 8 days.
7. £1914·006. **8.** £8099. 0s. 4·9d. **9.** £4285. 19s. 7d.
10. £103. 12s. 1·78d. **11.** £2727. 15s. 10·8d. **12.** £2857. 3s. 1d.
13. £10. 5s. $0\frac{1}{2}$d. **14.** £400. 0s. 0d.

Exercise LIV., p. 198.

1. £212. 18s. 4d. **2.** £1320. 0s. 0d. **3.** £4. 7s. 9d.
4. £78. 2s. 6d. **5.** 5 : 4. **6.** £171. 1s. $10\frac{1}{2}$d.
7. £13. 0s. 4d. **8.** £1023. 6s. 8d. **9.** £1040. 0s. 0d.
10. £7. 1s. 9d. **11.** £5. 19s. 2d.

Exercise LVIII., p. 223.

1. AB = 2·435 ; 64°·2 ; 35°·2 ; 67°·8. **2.** 56°·7 ; 39°·8 ; 70°·8.
3. PQ = 5·449 ft. **4.** 53°·15.

Exercise LIX., p. 226.

1. x = 30·69 ; y = 30·69 ; z = 25. **2.** 622·2 ; 510·9 lbs. per sq. in.
3. 22·2 tons ; 21·56 tons. **4.** 60°. **5.** 68° 12′ ; 5·385.
6. 9·063 ; 4·226. **7.** 5 ; 53°·13. **8.** 9·433 ; 58°.
9. 96·59 ; 25·88. **10.** 46·98 ; 17·1. **11.** 45·315 ; 21·13.
12. r = 5·386 ; θ = 68° 12′ ; ϕ = 52° 54′.
13. x = 1·239 ; y = 1·477 ; z = 2·298.

Exercise LX., p. 233.

1. 1 : 2. **2.** 9250 sq. ft. **3.** 3·952.
4. 280 and 140 yds. **5.** £16. 16s. 5½d. **6.** 1575 links.
7. 1·97 ; 1·31 ft. **8.** 59·733. **9.** 36·52 sq. ft.
10. 12·56 sq. ft.; £282. 12s. **11.** 5566 **12.** 19·23 sq. ft.
13. 640 acres. **14.** 360.

Exercise LXI., p. 237.

1. 2225. **2.** 188·1. **3.** 10 r. 0 p. 19·6 sq. yds.
4. 44·85 acres. **5.** 5937. **8.** 19·9. **9.** 108200.
10. 144·2 yds. **11.** 625·25 sq. ft. **12.** 561·18 sq. ft.
13. £1. 0s. 2d. **14.** 84 sq. ft.

Exercise LXII., p. 240.

1. 27·49. **2.** 30·096. **3.** 15548 sq. ft. **4.** 450 sq. ft.
5. 37 a. 1 r. 23·8 p. **6.** 368·5 sq. yds. **7.** 10·74 yds.
8. 24600 sq. miles. **9.** 368·5 sq. yds. **10.** 1030111 sq. lks.

Exercise LXIII., p. 243.

1. $47\frac{3}{4}$. **2.** 24854·85 miles. **3.** 10·99 inches. **4.** 81·68 ft.
5. 15·912. **6.** 76·35 ins. **7.** 72 ins. **8.** 100·9 miles.
9. 17°·19. **10.** 68·2647 ft. **11.** 6126·12 sq. yds.

Exercise LXIV., p. 246.

1. 623·45 sq. in. **2.** 11·28 ins.
3. (i) 785400 ; 31740 sq. ft. (ii) 3141·6 ; 3205 ft.
4. 346·9 sq. yds. **5.** 15268·176 lbs.
6. 706·86 lbs. **7.** 11310 lbs.
8. 7·85 ; ·00196 ; ·109 ; ·00077 ; ·110 ; ·049 ; ·000078 ; ·0038.

Exercise LXV., p. 247.

1. 1884·96 sq. ft. **2.** 22·57. **3.** 218·6; 10930 sq. ft.
4. 172·97. **5.** 464·1. **6.** 1253·88.
7. 119·4 sq. ft. **8.** 859 sq. ft.

Exercise LXVI., p. 249.

1. 5·4 sq. ft. **2.** 84·82 sq. ins.
3. 14·14 sq. ft.; 214 sq. ft. **4.** 1·058 sq. yds.
5. 110 sq. ins. **6.** 2985·6 lbs. **7.** 150·8. **8.** 707 lbs.

Exercise LXVII., p. 260.

1. 1704. **3.** 286. **4.** 11·2 sq. ft. **5.** 1 r. 13·13 po.
6. 460·6 sq. ft. **7.** 6252 sq. ft. **8.** 2686·5 sq. ft. **9.** 587·7.
10. 1778 sq. ft. **11.** 335·7. **12.** 2397·4.

Exercise LXVIII., p. 268.

1. 1008 cub. in. **2.** 229·7 cub. ft. **3.** $1333\frac{1}{3}$ cub. yds.
4. 562600; 4·768 ft. **5.** 45360. **6.** 8 ft.
7. 418 sq. ft. **8.** 608 lbs. **9.** 25532·935; 4294895·4.
10. 23·279 cub. ft.; 1·806 tons. **11.** 543·06 tons.
12. 6·191 ft.; 9000 lbs. **13.** 69·53 lbs.
14. 50 hours 50·4 mins. **15.** (*a*) 1·51; (*b*) 12 tons.
16. 68·9858. **17.** 54·427 cub. ft. **18.** 2037·1094 cub. ft.
19. 5·184 ins. **20.** £3. 4s. 11d. **21.** £4. 1s. 8d.
22. 1 ft. 3·04 ins. **23.** 2·588 cub. ft.; 1296 lbs.
24. 9·6 cub. ft.; 2·1 tons. **25.** 5·269 cub. ft.; 305·6 lbs.
26. 3·9 cub. ft.; ·537. **27.** 2018 lbs.
28. 341·5 lbs. **29.** 687·5 lbs.

Exercise LXIX., p. 273.

1. 4021·248 sq. ft.; 16084·992 cub. ft. **2.** 881·06.
3. 10·6 tons. **4.** 157080 cub. ft. **5.** 17610 lbs.
6. 8·767 ft. **7.** 4286 yds. **8.** 159 ft.; 1058·71 tons.
9. 19·8; 8·9 ins. **10.** 7 ft. 1·75 ins. **12.** 260·6 lbs.
13. 393 lbs.; $61\frac{6}{7}$ lbs. **14.** 1570·8 sq. ft.

Exercise LXX., p. 275.

1. 1018 lbs. **2.** 905 lbs. **3.** 10·8 inches.
4. 4·436 cub. ft.; ·97 ton. **5.** 3311·65 cub. in.; 960·28 lbs.
6. 326 cub. in.; 94·54 lbs. **7.** 111·9 lbs. **8.** 5·46 tons.
9. 1082·41 lbs. **10.** 2375·97 lbs. **11.** 19·157 sq. ft.
12. 52·13 lbs. **13.** 23 cwts. $8\frac{1}{4}$ lbs. **14.** 119·81 lbs.
15. 1805 yds. 1 ft. 2 in. **16.** 8 lbs. 15 oz. **17.** 31·386 lbs.
18. 28469 lbs. **19.** 839340·4 lbs.

Exercise LXXI., p. 280.

1. 379·82 cub. ft. **2.** 301·59 cub. ft. **3.** 121793·4 cub. ft.
4. 2412·75. **5.** 65·97. **6.** 1164·93 cub. ft.; 744 sq. ft.
7. $750\sqrt{3}$ cub. ft. **8.** 9 inches. **9.** 1·82 sq. ft.
10. 531·13 cub. ft. **11.** 649·4 cub. ft. **12.** 2664·2 galls.
13. 10·21 cub. ft. **14.** 1130·75 lbs. **15.** 127·234 days. **16.** 54·93.

Exercise LXXII., p. 283.

1. 10·193 inches. 2. 30·04 feet. 3. 1·852 ins.
4. 1·551 tons. 5. ·26334·08 cub. in. 6. $16\frac{1}{2}$ inches.
7. (*a*) 688·75 cub. ft.; 4290·9 galls.; (*b*) 3765·8 galls.; 3316 galls.
8. 2558·47. 9. 21912·66 sq. ft. 10. 3·6 ins.
11. 36·1 lbs. 12. The conical; ·05 pints. 13. 2·019 inches.
14. 87·66 tons. 15. ·205. 16. 191; ·844 oz.

Exercise LXXIII., p. 289.

1. 5 ft. per sec. 2. 10 ft. 3. $4\frac{1}{6}$ ft. 4. 695·5.
5. 5 to 4. 6. 3882 lbs. 7. 3456 lbs.; ·004 sec.
8. 2·15 tons. 9. 1000 lbs.

Exercise LXXIV., p. 291.

1. 3780000 ft.-lbs. 2. 423·36 ft.-lbs. 3. 3375 ft.-lbs.
4. 56520 ft.-lbs. 5. 475200 ft.-lbs. 6. 115000 ft.-lbs.

Exercise LXXV., p. 293.

3. 42·88 ft.-lbs.; 11·7 ft. per sec. 4. 48 tons.
5. $1 : \sqrt{2}$; 58733 ft.-lbs.

Exercise LXXVI., p. 295.

1. 4·2 radians; 10·9 ft. per sec. 2. ·2894 miles.
3. 2·2 radians; 13·09 ft. per sec. 4. 2·618 radians.
5. 4·4 radians. 6. 439·8 ft.

Exercise LXVII., p. 297.

1. 7·76 lbs. 2. 1·863 lbs. 3. 39·72 ft. per sec.; 7·94 radians.
4. 2661 lbs. 5. 17·46 lbs.

Exercise LXXVIII., p. 301.

1. 16; 136°; 152°. 2. 27; 33. 3. 9; 77°. 4. 11.
5. $8\frac{1}{2}$ ft. per sec.; 100°. 6. 17·32; 20. 7. 8·185.
8. 8·3 ft. per sec. per sec.; with force 5 angle of 19° 48′.

Exercise LXXIX., p. 305.

1. $23\frac{1}{2}$ tons.
2. (*a*) $15\frac{3}{4}$, $11\frac{1}{2}$ tons; (*b*) (1) $18\frac{1}{2}$; 10, (2) $14\frac{1}{2}$; $5\frac{1}{4}$, (*c*) $4\frac{7}{8}$; $2\frac{9}{16}$.
3. (*a*) 10, 16·85; 21·75, 15 tons. 4. 16 tons.
5. Jib, 11; stay, $6\frac{1}{4}$; back stay, 8; crane post, $3\frac{1}{4}$; balance weight, 6 tons.

Exercise LXXX., p. 308.

1. 86·6 ; 50 ; 100 ; 100. **2.** 7·84 units. **3.** 52·44 $\tan^{-1}\theta = -6{\cdot}4$.

4. 2·28 parallel to diagonal of square at a distance $\frac{3}{2}$ (diagonal of square) from it.

6. 15 and 10 units. **7.** 2·7 ; 2·34. **8.** 13 lbs.

9. 22·36. **10.** 57·3 ; 40·15.

Exercise LXXXI., p. 320.

1. $10\frac{1}{2}$, $9\frac{1}{3}$, $1\frac{2}{3}$, $-4\frac{5}{6}$, $-9\frac{1}{3}$, $-4\frac{4}{5}$, in the bars 1, 2, 3, 4, 5, 6 respectively.

3. Principal rafters, 5100 and 3800 respectively. Struts, 1350. Tie-beam, −4550. King post, −2200.

Exercise LXXXII., p. 324.

1. $2612\frac{1}{2}$ lbs.; 7″ from bottom edge ; 1950.

2. $1015\frac{2}{3}$ lbs.; 6″ from bottom edge ; 465.

3. $843\frac{3}{4}$ lbs.; $4\frac{1}{2}$″ from bottom edge ; 241·5.

4. 8·3 lbs.; 2·2″ ; 2·983. **5.** 439 lbs.; 175·9.

6. 1906·5 lbs.; 1151·4. **7.** 2786·5 lbs.; 4199·6. **8.** 1315·5 lbs.; 447·3.

Exercise LXXXIII., p. 334.

1. $R = 20{\cdot}7E - 165$. **2.** $E = {\cdot}4W + 4$.

3. $R = 23E - 3$; $F = 52E + 4$. **4.** ·315.

5. (i) $E = {\cdot}34R + {\cdot}83$; (ii) $E = {\cdot}35R + 5{\cdot}85$;
(iii) $E = {\cdot}7R + {\cdot}64$; $F = {\cdot}063R + {\cdot}43$; (iv) $E = {\cdot}385R + {\cdot}095$;
(v) $E = {\cdot}049R + {\cdot}13$; (vi) $R = 3{\cdot}08E - {\cdot}884$.

7. $\theta = 9{\cdot}8 \log T_2 - 18{\cdot}6$; $\theta = 9{\cdot}7 \log \frac{T_2}{T_1} - 1{\cdot}9$. **8.** $\theta = 19{\cdot}5 \log \frac{T_2}{T_1} - {\cdot}21$.

9. (i) $y = 177{\cdot}8e^{{\cdot}023t}$; (ii) $y = 178e^{-{\cdot}14t}$; (iii) $y = 249e^{-{\cdot}019t}$.

10. $y = 1{\cdot}5x + 2{\cdot}6$.

Exercise LXXXIV., p. 341.

1. $\frac{a}{b} = 1{\cdot}015$. **2.** $\frac{y}{x} = {\cdot}9025$. **3.** $\frac{x}{y} = {\cdot}7575$.

4. $x = 1{\cdot}3$. **5.** $x = 1{\cdot}3568$.

6. $x = -\frac{1}{3}$; $x = \frac{1}{2}$. **7.** Each part $= 5{\cdot}75$.

8. (i.) $x = 3$ (max.) ; (ii.) $x = 1$ (max.) ; $x = -1$ (min.) ; (iii.) $x = 2$ or 87 (max.) ; $x = -1$ or 3 (min.).

9. Each part $= \frac{n}{2}$. **10.** ·577d.

Exercise LXXXVI., p. 351.

1. 12·43. **2.** $\frac{4}{11}$ ft. **3.** 40 in.-lbs.
4. 200000 ft.-lbs. **5.** 25 ft.-tons. **6.** $31\frac{1}{2}$ ft.-lbs.
7. 26·366 ft.-lbs. **8.** ·00302 sq. in.; $11·8 \times 10^6$. **9.** 39062·5 lbs.
10. ·00896 in. **11.** 104·36 sq. in. **12.** 27·9. **13.** 33·8.

Exercise LXXXVII., p. 367.

1. ·02076. **2.** (i) ·00014; (ii) ·8461; 27th.
3. (i) ·0480; (ii) ·7235; (iii) ·3010.
4. (i) ·5537; (ii) − ·1388; (iii) $x=3·17$. **5.** ·725 in. **6.** ·1716.
7. (i.) ·0009451; ·00097; ·8106; (ii.) 109·2. **8.** 4·66. **9.** 524·9.
10. 11347500. **11.** 10021500. **12.** 4·731.
13. 45·68, 35·5, 28·89, 24·28, 20·87.
14. 53·59, 43·84, 38·95, 32·38, 28·72.
15. 66·66, 50·0, 40, $33\frac{1}{3}$, 28·57, 25. **16.** $a=$ ·0021; $c=$ ·017.
17. 4·276, 4·06, 3·043. **18.** (i) 22919 lbs.; (ii) 3602 lbs.; (iii) 6485 lbs.
19. 14005·6 in.-lbs. **20.** 2·54″ and 2·18″. **21.** 32360; 2·76″.
22. 1594924; 10·1. **23.** 6·47″; 1·69″. **24.** 6·2″.
25. 6·462, 6·504. **26.** 11·79; 64; 142·9. **27.** 5·6″.
28. 4 ins. **29.** 21035·5 ft.-lbs. **30.** 30×10^6.
31. ·8″; 1·97″. **32.** 99·3 lbs. **33.** 6·2″.
34. (*a*) 15·45 lbs.; (*b*) 20·38 ft.-lbs.; (*c*) 1·62.
35. 9·821; ·9676; ·0966; 98·21; 96·76; 96·61; actual velocity = 96·6.
36. $\frac{ds}{dt}=5+4·2t$; 26.

Examination Paper, p. 377.

1. 159·4; 3·468; 346·8; 2·865; 88·1; 1·65.
2. (*a*) 2·002; ·894. (*b*) ·892; 33·63. (*c*) 6·995.
3. (*a*) 29·92; 33·9; 14·74; 2120; 1·034. (*b*) 92·05; 92·5; 25 %. (*c*) ·06184″; 2·639.
4. (*a*) 2·38. (*b*) 2·638″; 5·026.
5. (*a*) 210. (*b*) 1588. (*c*) 1·443.
6. (*a*) 28·5 cub. ft. (*b*) 12790.
7. 71·8 ins.
8. (*a*) $y=·027+·178t$. (*b*) 4320; 76, 51, 61·7.
9. 27·8 millions; 0·5 million per annum.
10. (A) ·5236; 1·91 miles. (B) 9·425; 56·55 ft. per sec. (C) 849·6.

11. (A) 7·071; ·4242; ·5657; ·7071. (B) $x=1·748$; $y=2·083$; $z=1·268$.

12. 3·46; 2·57.

13. 3·06; 42°·5 N. of E.; ·946; 53° S. of W.; 3·79; 8°·5 E. of N.

14. 25·5; 255·082007; 82.

15. (1) Average value of y.—By ordinary rule 10·03; by Simpson's Rule 10·07. Rate of increase 2·015.

(2) Average value of y.—By ordinary rule 1·891; Simpson's Rule 1·902. Rate of increase ·1558.

(3) Average value of y.—By ordinary rule 1·795; Simpson's Rule 1·785. Rate of increase ·3007.

16. $x=2·18$; $x=2·11$; $x=2·22$.

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